2. Glimpses of the History of Mathematics in India

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Introduction

Starting from the representation of numbers, through the way of arriving at the solutions of indeterminate equations, to the development of sophisticated techniques in handling the infinite and the infititesimals, there has been a wide variation in the choice of working style amongst mathematicians of different cultures. By working style we mean the approach taken by mathematicians in formulating the problem, in internally visualizing the solution, in externally representing their consolidated understanding, and so on.

Most of the mathematics in India starting from 5th century CE has been handed down in the form of highly compressed and cryptic verses, not to mention the aphoristic sytle adopted prior to this period. As the transmission of knowledge was primarily oral, these verses/aphorisms used to be memorized and passed on orally form generation to generation—traces of which can be seen even today in several parts of India. The Indian mathematicians were so adept in metrical composition that even infinite series expansion of trigonometrical functions have been presented in the form of beautiful verses, which sometimes have a *double entendre*. While on this topic, it may also be mentioned that the purpose for which the power series were arrived at in India (around 14th century) as well as the route taken by mathematicians to arrive at them, are quite different from the trajectory adopted by mathematicians in Europe a couple of centuries later.

The purpose of this article in not to exalt a particular tradition or a working style over the other, but to provide a bird's eye view of the origin and development of mathematics in India by citing several passages form the original sources so as to enable the reader to have a flavor of the working style of Indian mathematicians and the kind of practical or application-oriented mathematics they developed. This apart from giving a broad picture of development of mathematics in India, would also help them acquire a cross-cultural perspective that would enable them to have a better appreciation of the evolution of mathematics across different cultures.

Mathematics in India has a very long and hallowed history. *Śulbasūtras*, the oldest extant texts (prior to 800 BCE) explicitly state and make use of the so-called Pythagorean theorem apart from giving various interesting approximations to surds, in connection with the construction of altars and fire-places of different sizes and shapes. By the time of Āryabhaṭa (c. 499 CE), the Indian mathematicians were fully conversant with most of the mathematics that we currently teach at the elementary level in our schools, which includes the methods for extracting square root, cube root, and so on. Among other things, Āryabhaṭa also presented the differential equation of sine function in its finite-difference form and a method for solving linear indeterminate equation. The *bhāvanā* law of Brahmagupta (c.628) and the *cakravāla* algorithm described by Jayadeva and Bhāskarācārya (12th cent.) for solving quadratic indeterminate equation are some of the important landmarks in the evolution of algebra in India.

The Kerala School of Astronomy and Mathematics pioneered by Mādhava of Saṅgamagrāma (c. 1350)—stumbling upon the problem of finding the exact relationship between the arc and the corresponding chord of a circle, and problems associated with that—came very close to inventing what goes by the name of infinitesimal calculus today. Particularly, Mādhava seems to have blazed a trail by enunciating the infinite series for $\frac{\pi}{4}$ (the so-called Gregory-Leibniz series) and other trigonometric functions.[1] As mentioned earlier, it is quite interesting to note that almost all these discoveries are succinctly coded in the form of metrical compositions in Sanskrit. To the present day reader, having got so much accustomed to the use of symbols, it may even be difficult to imagine a recursion relation, or an infinite series, or for that matter the derivative of a trigonometric function to be couched in the form of chaste prose or charming poetry (sometimes with an intended pun). But amazingly, that is how it has been presented to us at least from the time of *Śulbasūtras*, most of which were supposed to have been composed by 5th century BCE, till late 19th century CE. In what follows we attempt to provide a flavor of this mathematics with plenty of quotations from the original source works.

For the sake of convenience, we divide the paper into three sections (leaving out the first one on introduction). Section 2 deals with Mathematics in Ancient India (prior to 5th century CE), which would be followed by the section on Mathematics in the Classical period (500–1350 CE). In Section 4 we will discuss Mathematics in the Medieval period (14th – 16th cent.) which is described as the Golden Age of Mathematics in India. Before we embark upon the details, it wouldn't be out of place to quote the beautiful verses of Mahāvīrācārya (c. 9th cent.) conveying the ubiquity of mathematics.

[1] Interesting proofs of these results are presented in the famous Malayalam text *Gaṇita-Yuktibhāṣā* (c. 1530) of Jyeṣṭhadeva (Sarma, 2009) as well as in the works of Saṅkara Vāriyar, who was a contemporary of Jyeṣṭhadeva.

Mahāvīrācārya, in order to impress upon the importance of the study of mathematics, right at the very beginning of his classical treatise *Gaṇita-sāra-saṅgraha*, eloquently puts forth the diverse disciplines in which mathematics finds its application:

> लौकिके वैदिके चापि तथा सामयिकेपि यः। व्यापारस्तत्र सर्वत्र सङ्ख्यानमुपयुज्यते॥
> कामतन्त्रेऽर्थशास्त्रे च गान्धर्वे नाटकेऽपि वा। सूपशास्त्रे तथा वैद्ये वास्तुविद्यादि वस्तुषु॥
> छन्दोलङ्कारकाव्येषु तर्कव्याकरणादिषु। कलागुणेषु सर्वेषु प्रस्तुतं गणितं परम्॥...
>
> बहुभिर्विप्रलापैः किं त्रैलोक्ये सचराचरे। यत्किञ्चिद्वस्तु तत्सर्वं गणितेन विना न हि॥

> Whether the dealings have to do with worldly affairs or spiritual matters or religious practices, enumeration is very much involved. In affairs related to love, in economics, in music, in drama, in cooking, in practicing medicine, in the fields like architecture, in using metrics, in [employing] figures of speech, in [composing] literature, in logic, in grammar, in arts, etc, the mathematics [that is going to be discussed] is extremely important. ...
>
> Why keep talking much? In all the three worlds consisting of living and non-living entities, whatever be the transaction, it cannot be executed without mathematics!

Mathematics in the Ancient period

Śulbasūtras

The Vedic priests had developed a class of manuals that would assist them in the construction of altars (called *Vedis*) used for performing sacrifices. These texts called *Śulbasūtras* primarily dealing with the geometry related to the design of the *Vedis*, are considered to be a part of a larger class of texts known as *Kalpasūtras*, which in turn are considered to be one of the six *Vedāṅgas*.[2] The word *śulba* stems from the root *śulb* which means 'to measure'. Since all the measurements were done using ropes or chords in the very early times—traces of which can be found in practice even today—it seems the word in due course was synonymously employed to refer to the chords themselves.

[2]The term *Vedāṅga* is used to refer to six branches of knowledge namely *śikṣā, vyākaraṇam, kalpaḥ, niruktam, jyotiṣam* and *chandaḥ*. In olden times, all these branches used to be studied by every Vedic priest either after completing his studies of the *Veda*, or simultaneously along with it.

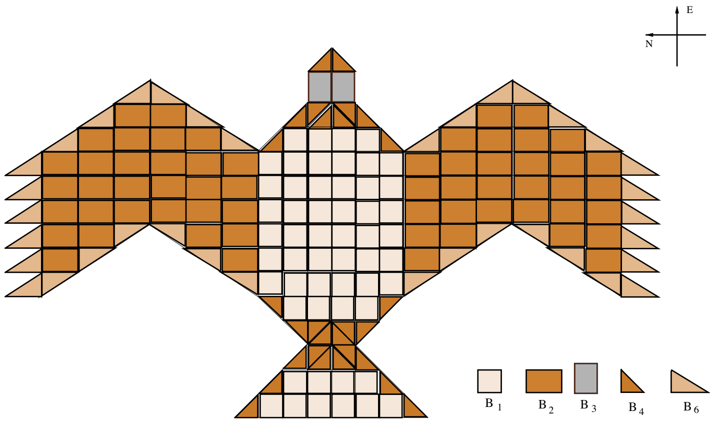

Figure 1: The first layer of the altar *Śyenaciti*.

Some of the geometrical constructions such as the *śyenaciti* (see Figures 1 and 2) prescribed by the Śulbakāras (the authors of the *Śulbasūtras*) are quite involved[3] and cannot be simply constructed without having a mastery over certain techniques that include the procedures for determining the east-west direction at a given location, for drawing straight lines that are at right angles to each other, for constructing a square whose side is surd times an integer, for finding the area of certain geometrical objects, and so on. We now proceed to discuss some of these basic tools as explained in *Śulbasūtras*.

Finding the cardinal directions

Having chosen the location at which the sacrificial altar is to be constructed, the first thing that needs to be done is the determination of the east-west direction at that location.[4] The procedure determining it is described as follows:

> समे शङ्कुं निखाय शङ्कुसम्मितया रज्वा मण्डलं परिलिख्य यत्र लेखयोः शङ्क्वग्रच्छाया निपतति तत्र शङ्कू निहन्ति सा प्राची।
>
> Fixing the gnomon (*śaṅku*) on level ground and drawing a circle with a cord measured by the gnomon, he fixes pins at points on the line (of the circum-

[3]In fact, there are a number of constraints that need to be fulfilled in the construction of *śyenaciti* such as, the number of bricks in each layer should be constant (200), the area of all the bricks put together must be equal to a specified number, and so on.

[4]In fact, this knowledge was a prerequisite for any kind of ritual prescribed in the Vedic literature and not necessarily the construction of altar or fireplace described in the *Śulbasūtras*.

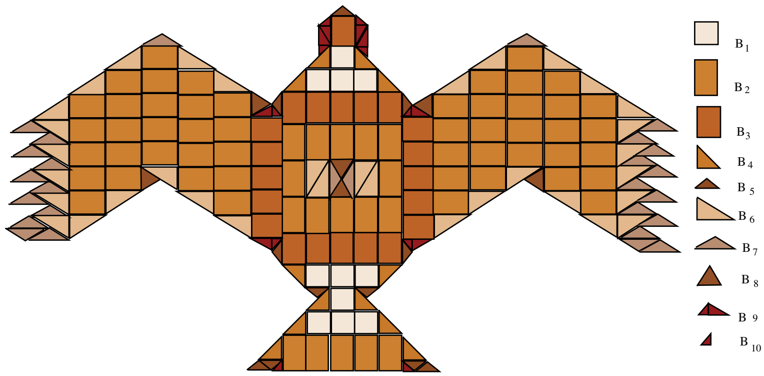

Figure 2: The second layer of the altar *Śyenaciti*.

ference) where the shadow of the tip of the gnomon falls. That is the east direction (*prācī*).

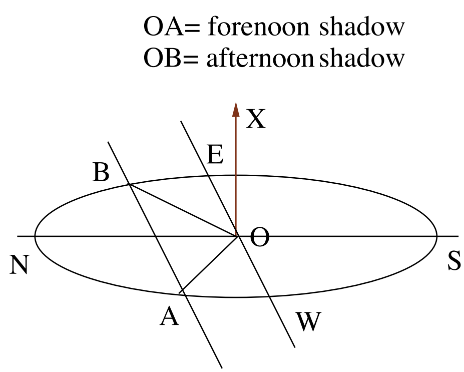

Figure 3: Determining the east-west line using *śaṅku*.

Asking the question as to why perform this experiment with *śaṅku*[5] in order to determine the direction, and not simply look at the sunrise and sunset and be with it, the commentator Mahīdhara observes:

[5]The term *śaṅku* refers to a very simple contrivance in the form of a rounded stick of a suitable length and height with a sharp tip at one of its edges. This has been extensively employed by Indian astronomers for conducting a variety of experiments to determine the cardinal directions at a given place, the latitude of the place and so on.

...तस्य उदयस्थानानां बहुत्वात् प्रतिदिनं भिन्नत्वात् अनियमेन प्राची ज्ञातुं न शक्या। तस्मात् शङ्कुस्थापनेन प्राचीसाधनमुक्तम्। दक्षिणायने चित्रापर्यन्तमर्कोऽभ्युदेति। मेषतुलासङ्क्रान्त्यहे प्राच्यां शुद्धायामुदेति। ततोऽर्कात् प्राचीज्ञानं दुर्घटम्।

Since the rising points are many, varying from day to day, the [cardinal] east point cannot be known [from the sunrise point]. Therefore it has been prescribed that the east be determined by fixing a *śaṅku*. ...

The theorem on the square of a diagonal

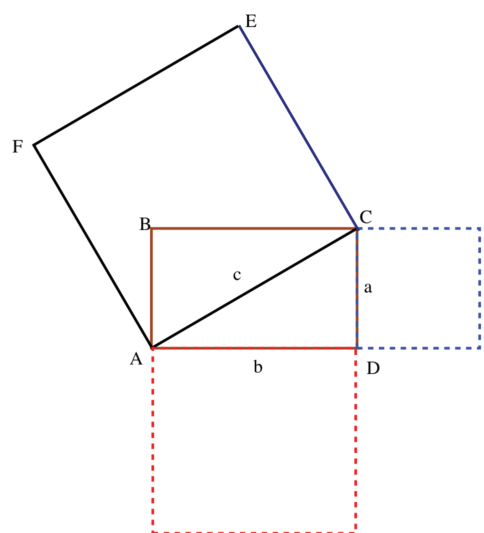

Figure 4: *Śulva* theorem: the theorem on the square of a diagonal.

The so called Pythagorean theorem (*bhujā-koṭi-karṇa-nyāya*) is given by Bodhāyana in his *Śulbasūtra* is as follows:

दीर्घचतुरश्रस्याक्ष्णया रज्जुः पार्श्वमानी तिर्यङ्मानी च यत्पृथग्भूते कुरुतः तदुभयं करोति।

The diagonal of a rectangle produces [an area] that is produced by the length and the breadth separately.

It may be noted (see Figure 4) that the actual enunciation of the theorem in *Śulbasūtras* is not with respect to the right-angled triangle but with respect to the sides and diagonals of squares and rectangles. It is well known that the famous mathematician-astronomer *Bhāskara* (12th century) gave an elegant proof of the theorem.[6] Yet another interesting proof (using dissectional method) is provided by Jyeṣṭhadeva in his *Yuktibhāṣā* (Sarma, 2009, pp.179-180).

To draw a square whose area is equal to n times a given square

Kātyāyana gives an interesting method for obtaining a square whose area is equal to the sum of the areas of a large number (say n) of squares.

> यावत्प्रमाणानि समचतुरश्राण्येकीकर्तुं चिकीर्षेत् एकोनानि तानि भवन्ति तिर्यक् द्विगुणान्येकत एकाधिकानि। त्र्यस्रिर्भवति तस्येषुस्तत्करोति।
>
> As many squares (of all side) as you wish to combine into one, the transverse line will be [equal to] one less than that; twice a side will be [equal to] one more than that. It will be a triangle (*tryasri*).[7] Its arrow (i.e., altitude) will do that.

Consider that there are n squares each of area a. It is desired that we obtain a square whose area is equal to the sum of all the n squares. The procedure given by Kātyāyana is to construct an isosceles triangle say ABC whose base is of length $(n-1)a$ and sides of length $\frac{(n+1)a}{2}$. It is said that the altitude of the triangle (AD) will give the side of a square ($\sqrt{n}a$) whose area will be na^2.
In Figure 5, $BD = \frac{1}{2}BC = (\frac{n-1}{2})a$. Considering the triangle ABD,

$$\begin{aligned} AD^2 &= AB^2 - BD^2 \\ &= \left[\frac{(n+1)a}{2}\right]^2 - \left[\frac{(n-1)a}{2}\right]^2 \\ &= \frac{a^2}{4}\left[(n+1)^2 - (n-1)^2\right] \\ &= na^2. \end{aligned}$$

[6] Making a note of this, Burger and Starbird in their recently published book (2010, p.233) observe:

> The proof presented here that was discovered by the Indian mathematician Bhaskara in the 12th century exemplifies aesthetics and beauty in mathematical arguments.

[7] Literally the word *tryasri* means a three-sided figure.

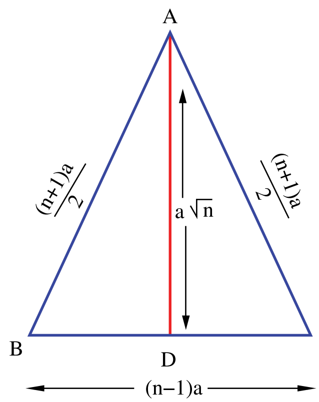

Figure 5: Scheme for drawing a square whose area to n times a given square proposed by Kātyāyana.

The prescription given above may look pretty straightforward and simple. But what is noteworthy is the amalgamation between geometry and algebra that it requires in order to come up with this prescription in its most 'general' form.[8]

Transforming a square into a circle

As mentioned earlier, the Vedic priests constructed altars of different sizes and shapes. In doing so, they had also imposed the constraint that altars of different shapes be of the same area. This naturally gives rise to the problem of transforming a square into a circle—over which mathematicians of all civilizations have struggled for ages. The prescription given by Bodhāyana for this problem is:

> चतुरश्रं मण्डलं चिकीर्षन् अक्ष्णयार्धं मध्यात् प्राचीम् अभ्यपातयेत्। यदतिशिष्यते तस्य सह तृतीयेन मण्डलं परिलिखेत्।
>
> Desirous of transforming a square into a circle, may the [length of the] semi-diagonal (*akṣṇayārdham*) be marked along the east direction starting from the centre. Whatever portion extends [beyond the side of the square], by adding one-third of that [to the semi-side of the square] may the circle be drawn.

[8]By 'general' form we mean the usage of *yāvat-tāvat* (as much-so much), which has been denoted by n in our explanation of the *sūtra*.

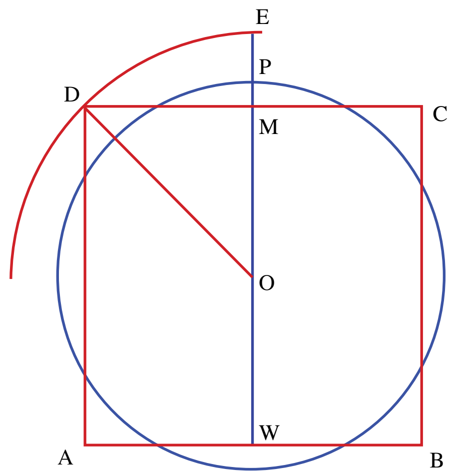

Figure 6: Constructing a circle whose area is same as that of a square.

In Figure 6, $AB = 2a,\ OP = r$ and $OD = a\sqrt{2}$. Hence $ME = a(\sqrt{2} - 1)$. Now the radius of the desired circle ($OP = r$) is given by

$$\begin{aligned} r &= a + \frac{a}{3}(\sqrt{2} - 1) \\ &= \frac{a}{3}(2 + \sqrt{2}). \end{aligned}$$

The above expression for radius involves finding the value of $\sqrt{2}$.

Value of $\sqrt{2}$ given in *Śulbasūtras*

The following *sūtra* given by the *Śulbakāras* (Bodhāyana, Āpastamba and Kātyāyana) presents an interesting rational approximation to $\sqrt{2}$:

प्रमाणं तृतीयेन वर्धयेत्, तच्चतुर्थेन, आत्मचतुस्त्रिंशोनेन, सविशेषः।

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}. \tag{1}$$

It must be noted that we have introduced an 'approximate' symbol in the above equation and not that of 'equality'. This is to signify the fact that the *sūtra* quoted above clearly states that the value specified is only approximate (*saviśeṣaḥ*, literally 'with a distinction/speciality').

Before we move on to the next topic, it may simply be mentioned that several scholars have attempted to offer explanation as to how the *Śulbakāras* might have arrived at the above expression,[9] which gives the value 1.4142157... that is remarkably close to the actual value 1.4142136....

The Bakhshālī Manuscript

The Bakhshālī Manuscript, a compendium of rules and illustrative examples related to arithmetic and algebra, was incidentally discovered by a farmer in the year 1881 in the course of excavation at a village called Bakhshālī in the north-west frontier of India.[10] This manuscript, which contained about 70 folios in birchbark (not all in readable condition), is found to be written in *Śāradā* script. Though the author and period of composition are not known, scholars are of the view that this should have been composed anywhere between 4th–7th century CE, if not earlier.[11]

Among other things, the manuscript presents certain interesting problems involving indeterminate equations along with their solutions. We present an example below.

Example: A jewel is sold among five merchants together. The price of the jewel is equal to half the money possessed by the first together with the moneys possessed by the others, or $\frac{1}{3}$rd the money possessed by the second together with the moneys possessed by the others, or $\frac{1}{4}$th the money possessed by the third together with the moneys possessed by the others, or $\frac{1}{5}$th the money possessed by the fourth together with the moneys possessed by the others, or $\frac{1}{6}$th the money possessed by the fifth together with the moneys possessed by the others. Find the cost of the jewel, and the money possessed by each merchant.[12]

Solution: If m_1, m_2, m_3, m_4, m_5 be the money possessed by the five merchants, and p be the price of the jewel, then the given problem may be represented as

[9]See for instance, the insightful article by Henderson (2000), and the one by Dani (2010), with incisive remarks.

[10]This place Bakhshālī is about 50 miles from Peshawar (currently in Pakistan).

[11]For erudite discussions on this issue see the introduction in (Sarasvati, Svami Satya Prakash & Jyotishmati, 1979), and an exclusive chapter devoted to this in (Hayashi, 2005).

[12]The statement of the problem commences as follows (Sarasvati, Svami Satya Prakash, & Jyotishmati, 1979, p.30):

> पञ्चानां वणिजां मध्ये मणिर्विक्रीयते किल। तत्रोक्ता मणिविक्रीत्रा मणिमूल्यं कियद्भवेत्॥ ...अर्धं त्रिभाग पादांश पञ्चभाग षडंश च।

Here it may be mentioned that though the solution to the problem is available in greater detail, the statement as such is not fully decipherable from the manuscript (see for instance, (Hayashi, 2005, pp. 174-175), and hence what has been presented above is a partially—yet faithfully—re-constructed version of it (see (Srinivasiengar, 1967, pp.39-40)) by gathering the information available in bits and pieces.

$$\begin{aligned}\frac{1}{2}m_1 + m_2 + m_3 + m_4 + m_5 &= m_1 + \frac{1}{3}m_2 + m_3 + m_4 + m_5 \\ &= m_1 + m_2 + \frac{1}{4}m_3 + m_4 + m_5 \\ &= m_1 + m_2 + m_3 + \frac{1}{5}m_4 + m_5 \\ &= m_1 + m_2 + m_3 + m_4 + \frac{1}{6}m_5 \\ &= p.\end{aligned}$$

Hence we have

$$\frac{1}{2}m_1 = \frac{2}{3}m_2 = \frac{3}{4}m_3 = \frac{4}{5}m_4 = \frac{5}{6}m_5 = q \text{ (say)}.$$

Substituting this in any of the previous equations we get $\frac{377}{60}q = p$. For integral solutions we have to take $p = 377\, r$ and $q = 60\, r$, where r is any integer. In fact, the answer provided in Bakhshālī manuscript is $p = 377$ and $m_1, m_2, m_3, m_4, m_5 = 120, 90, 80, 75, 72$ respectively.

Mathematics in the Classical Age

Starting with Āryabhaṭa in the 5th century, and extending upto Nārayaṇa Paṇḍita of the 14th century, the Indian mathematicians have blazed a trial in the study of several branches of mathematics that include obtaining recurrence relation for the construction of sine table, finding solutions to indeterminate equations of the first and the second degree, obtaining rules for finding the sum of arithmetic and geometric progression, finding the sum of sums, construction of magic squares, and so on. Some of the prominent mathematicians (most of them astronomers as well) who belonged to this period include Bhāskara I (c. 600), Brahmagupta (c. 628), Mahāvīra (c. 850), Pṛthūdaka (c. 860), Muñjāla (c. 932), Śrīpati (c. 1039), Bhāskara II (c. 1150), and Nārāyaṇa Paṇḍita (c. 1350). Due to the constraint on the length of the article, here we confine our discussion only to a select few topics listed above.

Solutions to indeterminate equations

The problem of finding integral solutions to indeterminate equations of the first and the second order has been of considerable interest to Indian mathematicians and astronomers of this age. An explicit algorithm for finding the general integral solution of the first order indeterminate equation of the form

$$ax + by = c, \tag{2}$$

called *kuṭṭaka* in Indian mathematics—popularly known as Diophantine equation—is found in *Āryabhaṭīya*. Āryabhaṭa as usual has been very cryptic and has presented the algorithm in just two verses (32 and 33 of *Gaṇitapāda*). However, it is interesting to note that Bhāskara I provides almost 30 examples in his commentary to illustrate the application of *kuṭṭaka* method prescribed by Āryabhaṭa. This clearly mirrors the need that was felt for solving such problems that occur in real life as well as in the context of solving certain problems in astronomy.

While Āryabhaṭa confined himself to dealing with first order indeterminate equation, Brahmagupta, a brilliant mathematician, who live about a century and a quarter later has attempted to solve a much harder problem of solving quadratic indeterminate equation of the form

$$Dx^2 + 1 = y^2, \tag{3}$$

where D is a positive integer that is not a perfect square. The principle invoked by Brahmagupta in solving equations of the above form has been referred to as *bhāvanā*. A detailed exposition of the *bhāvanā* principle, and the significant role it plays in modern algebra and number theory has been nicely brought out by Datta in one of his recent articles (2010). The solution developed by Brahmagupta has been improved upon by later Indian algebraists of whom special mention may be made of Jayadeva (early 11th century?) and Bhāskara II. The improved algorithm known as *cakravāla* has been illustrated by Bhāskara II by taking difficult[13] numerical cases like $D = 61$ and $D = 67$. A lucid explanation of Bhāskara's *Cakravāla* algorithm given in his *Bījagaṇita* (12th century), and its efficiency over Brounker-Wallis-Euler and Lagrange algorithm (17th century and 18th century) along with a numerical example, has been provided by Sriram in one of his recent articles (2005).

Sum of series & Sum of sums

Āryabhaṭa (c. 499 CE), in the *Gaṇitapāda* of *Āryabhaṭīya*, deals with a general arithmetic progression in verses 19–20. Following this, e gives the sum of the squares and cubes of natural numbers in verse 22:

[13] Difficult because, the smallest positive integer solution for the case $D = 61$ happens to be $x = 226153980$ and $y = 1766319049$.

सैकसगच्छपदानां क्रमात् त्रिसंवर्गितस्य षष्ठोऽशः।
वर्गचितिघनः स भवेत् चितिवर्गो घनचितिघनश्च॥

The product of the three quantities, the number of terms plus one, the same increased by the number of terms, and the number of terms, when divided by six, gives the sum of squares of natural numbers (*varga-citi-ghana*). The square of the sum of natural numbers gives the sum of the cubes of natural numbers (*ghana-citi-ghana*).

In other words,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{4}$$

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + n^3 &= [1 + 2 + 3 + \ldots + n]^2 \\ &= \left[\frac{n(n+1)}{2}\right]^2. \end{aligned} \tag{5}$$

Construction of sine-tables

The Indian astronomers took upon themselves the task of constructing sine-tables as accurately as possible, for their procedure for finding planetary positions—which in turn was crucial in the making the calendar, called *pañcāṅga*—was filled with sine and cosine functions. For constructing the sine-table, to be more precise the Rsine-table called *paṭhita-jyā*,[14] the circumference of a circle is divided into $21600'$ and usually the Rsines are tabulated for every multiple of $225'$, thus giving 24 tabulated Rsines in a quadrant. Using the value of $\pi \approx \frac{62832}{20000} = 3.1416$, given by Āryabhaṭa, the value of the radius turns out to be $3437'\ 44''\ 19'''$. This value, which is correct to seconds, was usually approximated to $3438'$.[15]

In the *Gītikā-pāda* of *Āryabhaṭīya* (verse 12), we find the following verse[16] that gives a table of Rsine-differences (the first differences of the values of trigonometric sines expressed in arcminutes):

मखि भखि फखि धखि णखि ञखि
ङखि हस्झ स्ककि किष्ग श्घकि किघ्व।

[14]In the Indian astronomical-mathematical treatises, the sine and cosine values were specified in minutes of arc and not in radians. The notation 'Rsine' is used to mark this distinction.

[15]Using a more accurate value of π, Mādhava (c. 1340–1420) gave the value of the radius correct to the thirds as $3437'\ 44''\ 48'''$ which in *Kaṭapayādi* notation is given by *devo-viśvasthalī-bhṛguḥ*.

[16]This verse is perhaps the most terse verse in the entire Sanskrit literature that the author of the paper has ever come across. Only after several trials would it be ever possible to read the verse properly, let also deciphering its content.

> घ्लकि किग्र हक्य धकि किच
> स्ग श्झ ङ्व क्ल प्त फ छ कलार्धज्याः ॥
>
> 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, and 7—these are the Rsine-differences [at intervals of 225′ of arc] in terms of the minutes of arc.

Incidentally it may be noted that the values presented here are correct to minutes, and this was perhaps the first ‘sine-table’ ever constructed in the history of mathematics.[17] How did Āryabhaṭa arrive at the above table?

In *Gaṇitapāda* (verse 12) Āryabhaṭa gives an ingenious method of computing the Rsine-differences, making use of the important property that the second-order differences of Rsines are proportional to the Rsines themselves:

> प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम्।
> तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि ॥
>
> The first Rsine divided by itself and then diminished by the quotient will give the second Rsine-difference. The same first Rsine, diminished by the quotients obtained by dividing each of the preceding Rsines by the first Rsine, gives the remaining Rsine-differences.

Let $B_1 = R\sin\,(225'),\; B_2 = R\sin\,(450'),...,\; B_{24} = R\sin\,(90^\circ)$, be the twenty-four Rsines, and let $\Delta_1 = B_1,\; \Delta_2 = B_2 - B_1,...,\; \Delta_k = B_k - B_{k-1},...$ be the Rsine-differences. Then, the above rule may be expressed as[18]

$$\Delta_2 = B_1 - \frac{B_1}{B_1} \tag{6}$$

$$\Delta_{k+1} = B_1 - \frac{(B_1 + B_2 + \ldots + B_k)}{B_1} \qquad (k = 1, 2, \ldots, 23). \tag{7}$$

This second relation is also sometimes expressed in the equivalent form

$$\Delta_{k+1} = \Delta_k - \frac{(\Delta_1 + \Delta_2 + \ldots + \Delta_k)}{B_1} \qquad (k = 1, 2, \ldots, 23). \tag{8}$$

From the above it follows that

$$\Delta_{k+1} - \Delta_k = \frac{-B_k}{B_1} \qquad (k = 1, 2, \ldots, 23). \tag{9}$$

[17] First because, the tables of Hipparchus (now lost) and Menelaus, as well as those of Ptolemy are all tables of chords and not of half-chords, as in the case of the table given by Āryabhaṭa.

[18] Āryabhaṭa is using the approximation $\Delta_2 - \Delta_1 \approx 1'$.

Since $\bar{\text{A}}\text{ryabha}\d{t}\text{a}$ also takes $\Delta_1 = B_1 = R\sin(225') \approx 225'$, the above relations reduce to

$$\Delta_1 = 225' \qquad (10)$$

$$\Delta_{k+1} - \Delta_k = \frac{-B_k}{225'} \qquad (k = 1, 2, \ldots, 23). \qquad (11)$$

In his scholarly preface to a recently published volume, David Mumford (2010) describes the procedure given by Āryabhaṭa as "the discrete analog of the result that sine solves the harmonic equation $y'' + y = 0$".

Nārāyaṇa Paṇḍita's general formula for *Vārasaṅkalita*

In his *Gaṇita-kaumudī*, Nārāyaṇa Paṇḍita (c. 1356) gives the formula for the r^{th}-order repeated sum of the sequence of numbers 1, 2, 3, …, n (Dvivedi, 1936, p.123):

> एकाधिकवारमिताः पदादिरूपोत्तरा पृथक् तेंऽशाः ।
> एकाद्येकचयहरास्तद्घातो वारसङ्कलितम् ॥
>
> The *pada* (number of terms in the sequence) is the first term [of an arithmetic progression] and 1 is the common difference. Take as numerators [the terms in the AP] numbering one more than *vāra* (the number of times the repeated summation is to be made). The denominators are [terms of an AP of the same length] starting with one and with common difference one. The resultant product is *vāra-saṅkalita*.

Let

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = V_n^{(1)}. \qquad (12)$$

Then, Nārāyaṇa's result is

$$V_n^{(r)} = V_1^{(r-1)} + V_2^{(r-1)} + \ldots + V_n^{(r-1)} \qquad (13)$$

$$= \frac{[n(n+1)\ldots(n+r)]}{[1.2\ldots(r+1)]}. \qquad (14)$$

The Cow problem

A notable feature of Nārāyaṇa—a feature that he shares with Bhāskarācārya, the author of *Līlāvatī*—is that he presents several interesting examples drawn from day to day life, that would not only be appealing to the reader, but also impress upon him the importance of the topic that is being discussed. After presenting the formula for finding the sum of sums, Nārāyaṇa illustrates the use of the above formula by choosing an interesting problem related to the estimate of population of cow.

> प्रतिवर्षं गौः सूते वर्षत्रितयेन तर्णकी तस्याः।
> विद्वन् विंशतिवर्षैः गौरेकस्याश्च सन्ततिं कथय॥
>
> A cow gives birth to a [she] calf every year [and] their calves themselves [begin giving birth], in 3 years time. O learned, tell the number of progeny produced by a cow in 20 years.

Recalling

$$
\begin{aligned}
V_n^{(0)} &= 1 + 1 + \ldots + 1 = n \\
V_n^{(1)} &= V_1^{(0)} + \ldots + V_n^{(0)} = 1 + 2 + \ldots + = \frac{n(n+1)}{2} \\
V_n^{(2)} &= V_1^{(1)} + V_2^{(1)} + \ldots + V_n^{(1)} = \frac{n(n+1)(n+2)}{1.2.3}
\end{aligned}
$$

For the sake of convenience we represent the solution of the problem in the form of a table (see Table 1).

Contribution of the Kerala School

The Kerala School, pioneered by Mādhava (c. 1340–1420) and followed by illustrious mathematicians and astronomers like Parameśvara, Dāmodara, Nīlakaṇṭha, Acyuta and others, by introducing several new ideas and techniques built an elaborate mathematical edifice that forms part of what is known as Calculus today. Our aim in this section, is to provide a glimpse[19] of some of the brilliant discoveries—such as the sum of infinite geometric series, the infinite series for $\frac{\pi}{4}$, its fast convergent approximations, and so on—that were made by the mathematicians of the Kerala School, anticipating some of the developments in Europe made almost two centuries later. Before we embark upon the details it may be mentioned that a systematic exposition of the work of the Kerala School, is to be found in the famous Malayalam work *Gaṇita-yukti-bhāṣā* (Rationales in Mathematical Astronomy) (Sarma, 2009) composed by Jyeṣṭhadeva (c. 1530)—a disciple of Dāmodara and junior to Nīlakaṇṭha.

[19] For a detailed exposition of the Development of Calculus in India, the readers are referred to the article by Ramasubramanian and Srinivas (2010).

Year	1^{st}gen.	2^{nd}gen.	3^{rd}gen.	4^{th}gen.	5^{th}gen.	6^{th}gen.	7^{th}gen.
1	1						
2	1						
3	1						
4	1	$V_1^{(0)}$					
5	1	$V_2^{(0)}$					
6	1	$V_3^{(0)}$					
7	1	$V_4^{(0)}$	$V_1^{(1)}$				
8	1	$V_5^{(0)}$	$V_2^{(1)}$				
9	1	$V_6^{(0)}$	$V_3^{(1)}$				
10	1	$V_7^{(0)}$	$V_4^{(1)}$	$V_1^{(2)}$			
11	1	$V_8^{(0)}$	$V_5^{(1)}$	$V_2^{(2)}$			
12	1	$V_9^{(0)}$	$V_6^{(1)}$	$V_3^{(2)}$			
13	1	$V_{10}^{(0)}$	$V_7^{(1)}$	$V_4^{(2)}$	$V_1^{(3)}$		
14	1	$V_{11}^{(0)}$	$V_8^{(1)}$	$V_5^{(2)}$	$V_2^{(3)}$		
15	1	$V_{12}^{(0)}$	$V_9^{(1)}$	$V_6^{(2)}$	$V_3^{(3)}$		
16	1	$V_{13}^{(0)}$	$V_{10}^{(1)}$	$V_7^{(2)}$	$V_4^{(3)}$	$V_1^{(4)}$	
17	1	$V_{14}^{(0)}$	$V_{11}^{(1)}$	$V_8^{(2)}$	$V_5^{(3)}$	$V_2^{(4)}$	
18	1	$V_{15}^{(0)}$	$V_{12}^{(1)}$	$V_9^{(2)}$	$V_6^{(3)}$	$V_3^{(4)}$	
19	1	$V_{16}^{(0)}$	$V_{13}^{(1)}$	$V_{10}^{(2)}$	$V_7^{(3)}$	$V_4^{(4)}$	$V_1^{(5)}$
20	1	$V_{17}^{(0)}$	$V_{14}^{(1)}$	$V_{11}^{(2)}$	$V_8^{(3)}$	$V_5^{(4)}$	$V_2^{(5)}$
Sum	20	153	560	1001	762	210	8

Table 1: The population of cow in 20 years.

Sum of an infinite geometric series

While deriving an interesting approximation for an arc of a circle in terms of the *jyā* (Rsine) and the *śara* (Rversine),[20] Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya* presents a de-

[20]Considering a circle of radius R, if $s = R\theta$ is the arc of a circle, subtending an angle θ (in radians) at the centre, then

$$\begin{aligned} jy\bar{a}(s) &= R\sin\theta \\ \acute{s}ara(s) &= \text{Rvers}\,\theta \;=\; R(1-\cos\theta). \end{aligned}$$

tailed explanation of how to sum an infinite geometric series. The specific series that arises in this context is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}\,. \tag{15}$$

At the outset, Nīlakaṇṭha poses a very important question (Śāstrī, 1930, p.106):

> **कथं पुनः तावदेव वर्धते तावद्वर्धते च ?**
>
> How do you know that [the sum of the series] increases only upto that [limiting value] and that it certainly increases upto that [limiting value]?

Proceeding to answer the above question, Nīlakaṇṭha first obtains the sequence of results

$$\begin{aligned}\frac{1}{3} &= \frac{1}{4} + \frac{1}{(4.3)}\,, \\ \frac{1}{(4.3)} &= \frac{1}{(4.4)} + \frac{1}{(4.4.3)}\,, \\ \frac{1}{(4.4.3)} &= \frac{1}{(4.4.4)} + \frac{1}{(4.4.4.3)}\,,\end{aligned}$$

and so on, which leads to the general result

$$\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n\left(\frac{1}{3}\right). \tag{16}$$

Nīlakaṇṭha then goes on to present the crucial argument: As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$ (as given by RHS of the above equation), becomes extremely small, but never zero. Only when we take all the terms of the infinite series together do we obtain the equality expressed in (15).

Mādhava series for π

The infinite series for $\frac{\pi}{4}$ enunciated by Mādhava in the form of a verse,[21]

---

[21] This verse is quoted by Śaṅkara Vāriyar in his commentary *Kriyākramakarī* on *Līlāvatī* (Sarma, 1975, p.379).

> व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
> त्रिशरादिविषमसङ्ख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात्॥
>
> The diameter multiplied by four and divided by unity [is found and saved]. Again the products of the diameter and four are divided by the odd numbers like three, five, etc., and the results are subtracted and added in order.

is the well known series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots \quad . \tag{17}$$

We shall now present the derivation of the above result as outlined in *Yuktibhāṣā* (Sarma, 2009 pp. 183–198). For this, let us consider the quadrant OP_0P_nS of the square circumscribing the given circle (see Figure 1) of radius r. Divide the side P_0P_n into n equal parts (n very large). The resulting segments P_0P_i's ($i = 1, 2, \dots, n$) are known as the *bhujās* and the line joining its tip and the centre OP_i's are known as *karṇas*. The points of intersection of these *karṇas* and the circle are denoted by A_i. The *bhujās* P_0P_i, the *karṇas* k_i and the east-west line OP_0 form right-angled triangles whose hypotenuses are given by

$$k_i^2 = r^2 + \left(\frac{ir}{n}\right)^2 . \tag{18}$$

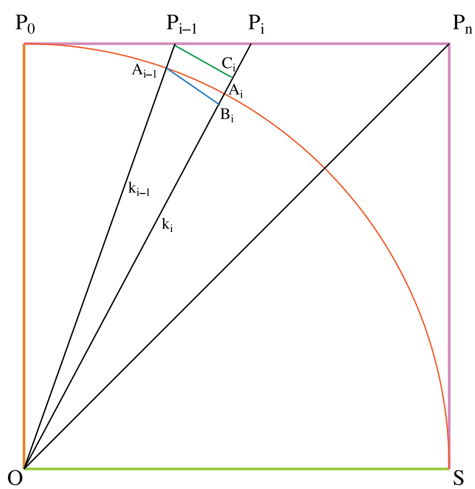

Figure 7: Geometrical construction used in the proof of the infinite series for π.

Considering two successive *karṇas*—ith and the previous one as shown in the figure—and the pairs of similar triangles, $OP_{i-1}C_i$ and $OA_{i-1}B_i$ and $P_{i-1}C_iP_i$ and P_0OP_i, it can be shown that

$$A_{i-1}B_i \;=\; \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right). \tag{19}$$

Now the text presents the crucial argument: When n is large, the Rsines $A_{i-1}B_i$ corresponding to different arc-bits $A_{i-1}A_i$ can be taken as the arc-bits themselves. Thus, $\frac{1}{8}$th of the circumference of the circle can be written as the sum of the contributions given by (19).

$$\frac{C}{8} \approx \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0k_1}\right) + \left(\frac{r^2}{k_1k_2}\right) + \cdots + \left(\frac{r^2}{k_{n-1}k_n}\right)\right]. \tag{20}$$

It is further argued in the text that the denominators $k_{i-1}k_i$ may be replaced by the square of either of the *karṇas* i.e., by k_{i-1}^2 or k_i^2 since the difference is negligible. Thus (20) may be re-written in the form

$$\begin{aligned}\frac{C}{8} &= \sum_{i=1}^{n} \frac{r}{n}\left(\frac{r^2}{k_i^2}\right)\\ &= \sum_{i=1}^{n}\left(\frac{r}{n}\right)\left(\frac{r^2}{r^2+\left(\frac{ir}{n}\right)^2}\right)\\ &= \sum_{i=1}^{n}\left[\frac{r}{n} - \frac{r}{n}\left(\frac{\left(\frac{ir}{n}\right)^2}{r^2}\right) + \frac{r}{n}\left(\frac{\left(\frac{ir}{n}\right)^2}{r^2}\right)^2 - \ldots\right]\end{aligned} \tag{21}$$

In the series expression for the circumference given above, factoring out the powers of $\frac{r}{n}$, the summations involved are that of even powers of the natural numbers. It was known to Indian mathematicians that

$$\sum_{i=1}^{n} i^k \;\approx\; \frac{n^{k+1}}{k+1}\;. \tag{22}$$

Now, using the estimate (22) for these sums when n is large, we arrive at the result[22]

$$\frac{C}{8} = r\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \tag{23}$$

[22] In modern terminology, the above derivation amounts to the evaluation of the following integral

$$\frac{C}{8} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{r}{n}\right)\left(\frac{r^2}{r^2+\left(\frac{ir}{n}\right)^2}\right) = r\int_0^1 \frac{dx}{1+x^2}.$$

which is same as (17), the well known series for $\frac{\pi}{4}$.

Discussion

Some of the novel insights and techniques of handling the infinitesimals and the infinite sum developed by the Kerala school of mathematicians, was first brought to the notice of the western scholarship by Charles Whish (1834) in early nineteenth century.[23] Somehow this seems to have gone unnoticed among the historians of mathematics for more than a century, that as late as 1940s the renowned historians like Carl Boyer (1949, pp.61-62) make infelicitous remarks:

> They (Hindus) delighted more in the tricks that could be played with numbers than in the thoughts the mind could produce . . . The Pythagorean problem of the incommensurable, which was of intense interest to Greek geometers, was of little import to Hindu mathematicians, who treated rational and irrational quantities, curvilinear and rectilinear magnitudes indiscriminately.

However, subsequent studies have led to a somewhat different perception of the Indian contribution as may be gleaned from the following quotation from a recent work on the history of mathematics (Hodgekin, 2005, p.168):

> We have here a prime example of two traditions whose aims were completely different. The Euclidean ideology of proof which was so influential in the Islamic world had no apparent influence in India (as al-Biruni had complained long before), . . . To suppose that some version of 'calculus' underlay the derivation of the series must be a matter of conjecture.
> The single exception to this generalization is a long work, much admired in Kerala, which was known as *Yuktibhāṣā*, by Jyeṣṭhadeva; this contains something more like proofs—but again, . . .

In the recent past there has been an attempt to assess the Indian contribution to the development of calculus by several scholars. To the question whether the Kerala school invented calculus, while some have answered in the affirmative, the others have some

[23]Though this remarkable paper of Whish got published only in 1934, from the notings made by John Warren in his *Kālasaṅkalita* (1825), we understand that Whish had communicated his findings to some of the senior officers like George Hyne as early as 1825. However, the views maintained by Whish that the infinite series were found by 'Natives' themselves were not received favorably. George Hyne in one of his correspondence observes: "the Hindus never invented the series; it was communicated with many others, by Europeans, to some learned Natives in modern times. . . the pretensions of the Hindus to such a knowledge of geometry, is too ridiculous to deserve refutation". For more details of the episode see Sarma et.al (2010).

reservations in accepting this position (Katz, 1995; Raju, 2001; Bressoud, 2002; Divakaran, 2007)—as easily evident from Hodgekin's observation quoted above.[24] In this connection, we would like to present a few facts before the readers.

Indian work on calculus 'primarily' stems from solving problems in Astronomy. To be more precise, it got developed as a part of the continuous endeavor on the part of astronomers to improve the precision of their calculations that involves sine and cosine functions, and their derivatives. All the astronomer-mathematicians starting from Āryabhaṭa to Mādhava had a conception of planetary model wherein they had to deal with only circles, and sine and cosine functions (whose differentials repeat after two orders). This, along with the fact that the pursuit of mathematics in India was 'primarily' calculation or application-oriented, explains why arbitrary functions and curves were not considered by Indian mathematicians.

However, recalling the fact that there are essentially three founding pillars on which the edifice of calculus rests upon—one, splitting the curve into infinitesimal parts, two, locally linearizing them, and three, summing up their 'infinite' infinitesimal contributions—and the fact that all the three are found in their full blown form in the derivation of Mādhava series for $\frac{\pi}{4}$ as presented above, and also the fact that "the muse of mathematics can be wooed in many different ways",[25] we leave it to the readers to judge for themselves, 'how' and 'where' to place the contribution of Mādhava,[26] in the context of narrating the grand story of the discovery of calculus.

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[24]Interestingly, there are a few scholars who have come up with models for the transmission of calculus from India to Europe (Joseph & Almedia,2007; Raju, 2007).

[25]Towards the end of his review of the book *Mathematics in India* (Plofker, 2008) David Mumford, the renowned mathematician and Fields medalist observes:

> Rigorous mathematics in the Greek style should not be seen as the only way to gain mathematical knowledge. ... the muse of mathematics can be wooed in many different ways and her secrets teased out of her (Mumford, 2010).

[26]Though Mādhava has been the acknowledged fountainhead of the profound ideas that emerged from the Kerala School, unfortunately none of his works on mathematics per se, but for a couple of works in astronomy, are extant now. It is only from the quotations and citations made by the later astronomers and mathematicians of this School that we come to know of some of his brilliant contributions.

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