Glimpses of the History of Mathematics in India

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Introduction

- Different civilizations have wooed the muse of mathematics in different ways. However, as has been rightly pointed out:[1]

  > *Too many people still think that mathematics was born in Greece and more or less slumbered until the Renaissance.*

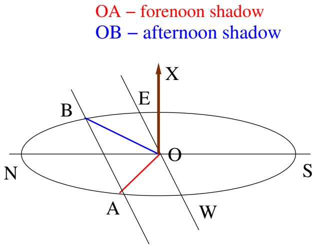
- Consequently, the general picture that emerged till recent times were quite incomplete and at times misleading.
- The present paper besides contributing (infinitesmally) towards correcting this picture, aims at providing a flavor of the origin and growth of mathematics in India.
- We attempt to do this by dividing the history of more than two millenia in into broadly three periods:
  - Mathematics in the Ancient Period (prior to 500 CE)
  - Mathematics during the Classical Age (500 – 1250 CE)
  - Mathematics in the Medieval Period (1250 – 1750 CE)

[1] David Mumford in his preface, to *Contributions to the History of Indian Mathematics*, Ed. C. S. Seshadri, HBA, Delhi 2010.

Mathematics in the Ancient Period

The *Śulbasūtra* method of finding the east-west direction

- The *Sulbasūtra* texts (composed ≈ 500 BCE) may be considered as manuals that assist the Vedic priests in the construction of altars (*vedis* or *citis*) used for performing sacrifices.
- The procedure for determining the east-west line (a pre-requisite for all constructions) is described thus:



समे शङ्कुं निखाय ‘शङ्कुसम्मितया रज्ज्वा’
मण्डलं परिलिख्य यत्र लेखयोः शङ्क्वग्रच्छाया
निपतति तत्र शङ्कू निहन्ति सा प्राची।

[*Kt. Su.* I 2]

Fixing a pin (or gnomon) on level ground and drawing a circle with a cord measured by the gnomon, . . . That is the *prācī*.

To transform a square into a circle

How did the *Sulvakāras* specify the value of $\sqrt{2}$?

- The following *sūtra* gives an approximation to $\sqrt{2}$:

  प्रमाणं तृतीयेन वर्धयेत्, तच्चतुर्थेन, आत्मचतुस्त्रिंशोनेन, सविशेषः।

  May the measure be increased by one-third ...

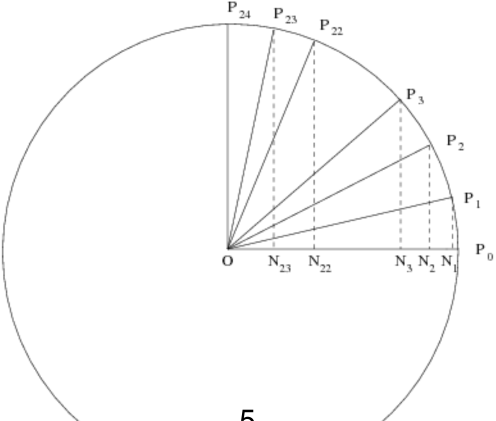
$$
\begin{aligned}
\sqrt{2} &= 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34} \qquad (1) \\
&= \frac{577}{408} \\
&= 1.4142156\ldots
\end{aligned}
$$

- How did the *Śulvakāras* arrive at the above expression?
- Several explanations have been offered over the last centuries.

Mathematics during the Classical Age

Construction of the sine-table (Āryabhaṭa's method)

- A quadrant is divided into 24 equal parts, so that each arc bit $\alpha = \frac{90}{24} = 3^\circ 45' = 225'$.
- Procedure for finding $R \sin i\alpha, (P_i N_i)$ $i = 1, 2, \ldots 24$ is given.
- The R sines of the intermediate angles are determined by interpolation (I order or II order).



Recursion relation for the construction of sine-table

Āryabhaṭīya's algorithm for constructing of sine-table

- In *Gaṇitapāda* (verse 12) Āryabhaṭa gives an ingenious method of computing the Rsine-differences:

  प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम्।
  तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि॥

- The content of the above verse translates to:

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin i\alpha - R\sin(i-1)\alpha - \frac{R\sin i\alpha}{R\sin\alpha}.$$

- What is noteworthy in the above equation is that it is the discrete version of the harmonic equation $y'' + y = 0$.
- Approximation used by Āryabhaṭa is $2(1-\cos\alpha) = \frac{1}{R\sin\alpha} = \frac{1}{225}$.
- While, $2(1-\cos\alpha) = 0.0042822$, $\frac{1}{225} = 0.00444444$.
- In the recursion relation provided by Nīlakaṇṭha we find $\frac{1}{225} \to \frac{1}{233.5} (= 0.0042827)$.

Recursion relation for the construction of sine-table

Āryabhaṭīya's algorithm for constructing of sine-table

- In the *Gītikā-pāda* of *Āryabhaṭīya* (verse 12), we find the following verse[2] that gives a table of Rsine-differences (expressed in arcminutes):

  मखि भखि फखि धखि णखि ञखि
  ङखि हस्झ स्ककि किष्ग स्घकि किघ्व।
  घ्लकि किग्र हक्य धकि किच
  स्ग श्झ ङ्व क्ल प्त फ छ कलार्धज्याः॥

  225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, and 7—these are the Rsine-differences [at intervals of 225′ of arc] in terms of the minutes of arc.

- In Āryabhaṭa's notation: म $\to$ 25; & खि $\to$ 200;

[2] This verse is perhaps the most terse verse in the entire Sanskrit literature that I have ever come across. Only after several trials would it be ever possible to read the verse properly, let also deciphering its content.

Saṅkalita & *Vārasaṅkalita*

Sum of series & Sum of sums

- The results

$$\sum_{r=1}^{n} r = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

and

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

were known to the authors of *Āryabhaṭīya*, *Gaṇitasārasaṅgraha* ...

- Now we consider sum of sums, for which we use the notation,

$$V_n^{(1)} = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$V_n^{(2)} = V_1^{(1)} + V_2^{(1)} + \ldots V_n^{(1)}$$

$$= \sum_{r=1}^{n} \frac{r(r+1)}{2} = \frac{n(n+1)(n+2)}{1.2.3}$$

Vārasaṅkalita

k^{th} order repeated sum of natural numbers (in *Gaṇitakaumudī*)

Nārayaṇa presents the formula in the following *Āryā*:

- एकाधिकवारमिताः पदादिरूपोत्तराः पृथक् तेऽम्शाः।
  एकाद्येकचयहराः तद्घातो वारसङ्कलितम्॥
  - पदादिरूपोत्तराः – $n, \; n+1, \; n+2, \ldots$
  - एकाधिकवारमिताः – limited to $k+1$ terms
  - एकाद्येकचयहराः – 1, 2, 3, ... are the divisors
  - तद्घातो वारसङ्कलितम् – the product is *vārasaṅkalita*

- The sum of sums

$$V_n^{(k)} = V_1^{(k-1)} + V_2^{(k-1)} + \ldots V_n^{(k-1)}$$

is stated to be

$$V_n^{(k)} = \frac{n(n+1)\ldots(n+k)}{1.2.3.\ldots k+1} = {}^{n+k}C_{k+1}$$

The Cow Problem

प्रतिवर्षं गौः सूते वर्षत्रितयेन तर्णकी तस्याः ।
विद्वन् विंशतिवर्षैः गौरेकस्याश्च सन्ततिं कथय ॥

A cow gives birth to a [she] calf every year [and] their calves themselves [begin giving birth], in 3 years time. O learned, tell the number of progeny produced by a cow in 20 years.

Recalling

$$
\begin{aligned}
V_n^{(0)} &= 1 + 1 + \ldots + 1 = n \\
V_n^{(1)} &= V_1^{(0)} + \ldots + V_n^{(0)} = 1 + 2 + \ldots + = \frac{n(n+1)}{2} \\
V_n^{(2)} &= V_1^{(1)} + V_2^{(1)} + \ldots + V_n^{(1)} = \frac{n(n+1)(n+2)}{1.2.3}
\end{aligned}
$$

The Cow Problem

Year	1^{st}gen.	2^{nd}gen.	3^{rd}gen.	4^{th}gen.	5^{th}gen.	6^{th}gen.	7^{th}gen.
1	1						
2	1						
3	1						
4	1	$V_1^{(0)}$					
5	1	$V_2^{(0)}$					
6	1	$V_3^{(0)}$					
7	1	$V_4^{(0)}$	$V_1^{(1)}$				
8	1	$V_5^{(0)}$	$V_2^{(1)}$				
9	1	$V_6^{(0)}$	$V_3^{(1)}$				
10	1	$V_7^{(0)}$	$V_4^{(1)}$	$V_1^{(2)}$			
11	1	$V_8^{(0)}$	$V_5^{(1)}$	$V_2^{(2)}$			
12	1	$V_9^{(0)}$	$V_6^{(1)}$	$V_3^{(2)}$			
13	1	$V_{10}^{(0)}$	$V_7^{(1)}$	$V_4^{(2)}$	$V_1^{(3)}$		
14	1	$V_{11}^{(0)}$	$V_8^{(1)}$	$V_5^{(2)}$	$V_2^{(3)}$		
15	1	$V_{12}^{(0)}$	$V_9^{(1)}$	$V_6^{(2)}$	$V_3^{(3)}$		
16	1	$V_{13}^{(0)}$	$V_{10}^{(1)}$	$V_7^{(2)}$	$V_4^{(3)}$	$V_1^{(4)}$	
17	1	$V_{14}^{(0)}$	$V_{11}^{(1)}$	$V_8^{(2)}$	$V_5^{(3)}$	$V_2^{(4)}$	
18	1	$V_{15}^{(0)}$	$V_{12}^{(1)}$	$V_9^{(2)}$	$V_6^{(3)}$	$V_3^{(4)}$	
19	1	$V_{16}^{(0)}$	$V_{13}^{(1)}$	$V_{10}^{(2)}$	$V_7^{(3)}$	$V_4^{(4)}$	$V_1^{(5)}$
20	1	$V_{17}^{(0)}$	$V_{14}^{(1)}$	$V_{11}^{(2)}$	$V_8^{(3)}$	$V_5^{(4)}$	$V_2^{(5)}$
Sum	20	153	560	1001	762	210	8

Mathematics in the Medieval Period

A very interesting approximation to π (correct to 11 decimal places)

- An interesting ratio for π is attirbuted to Mādhava (14th cent.)

$$\pi = \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592 \qquad \text{(correct to 11 places)}$$

- Two questions arise:
  1. Why was Mādhava interested in such accurate value?
  2. And, how did he obtain that?
- Answer to the latter: Mādhava obtained an infinite series[3] for π

$$Paridhi = 4 \times Vy\bar{a}sa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right),$$

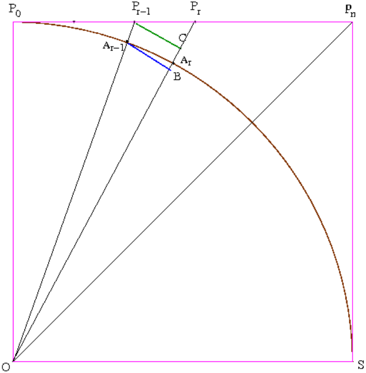
  and ingeniously handled it to obtain fast convergent approximations to get this accuracy.

[3] व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
त्रिशरादि विषमसङ्ख्याभक्तम् ऋणं स्वं पृथक् क्रमात् कुर्यात्॥

Infinite series for π

By considering similar triangles it can be shown that the Rsine corresponding to the *i*th arc-bit is given by,

$$A_{i-1}B_i = \left(\frac{r}{n}\right)\left(\frac{r^2}{k_i k_{i+1}}\right) \quad (2)$$



The text notes that, when the *khaṇḍas* $\left(\frac{r}{n}\right)$ become small, (or equivalently n becomes large), the Rsines can be taken as the arc-bits itself.

परिधिखण्डस्यार्धज्या $\rightarrow$ परिध्यंश

i.e., $A_{i-1}B_i \rightarrow A_{i-1}A_i$.

(local approximation by linear functions i.e., tangents/differentiation)

Infinite series for π

It can be shown that the problem boils down to finding the sum,

$$\begin{aligned}
\frac{C}{8} &= \sum_{i=1}^{n} \frac{r}{n}\left(\frac{r^2}{k_i^2}\right) \quad \textit{summming infinitesmals/integration} \\
&= \sum_{i=1}^{n}\left[\frac{r}{n} - \frac{r}{n}\left(\frac{k_i^2 - r^2}{r^2}\right) + \frac{r}{n}\left(\frac{k_i^2 - r^2}{r^2}\right)^2 - \ldots\right] \\
&= \left(\frac{r}{n}\right)[1 + 1 + \ldots + 1] \\
&\quad - \left(\frac{r}{n}\right)\left(\frac{1}{r^2}\right)\left[\left(\frac{r}{n}\right)^2 + \left(\frac{2r}{n}\right)^2 + \ldots + \left(\frac{nr}{n}\right)^2\right] \\
&\quad + \left(\frac{r}{n}\right)\left(\frac{1}{r^4}\right)\left[\left(\frac{r}{n}\right)^4 + \left(\frac{2r}{n}\right)^4 + \ldots + \left(\frac{nr}{n}\right)^4\right] \\
&\quad + \ldots \quad .
\end{aligned} \tag{3}$$

Kerala astronomers knew that $\sum_{i=1}^{n} i^k \approx \frac{n^{k+1}}{k+1}$, which immediately yields

$$\frac{C}{8} = r\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \tag{4}$$

Concluding Remarks

- Introducing history would make mathematics education more complete and enable to develop cross-cultural perspective.
- Through out my schooling (which of course was in India), I do not remember to have come across a single lesson talking about the Indian contribution to mathematics.[4]
- The examples presented above,
  - the prescription in *Śulbasūtra* texts for finding the cardinal direction, value of surds, etc.
  - the recurrence relation as well as the table given by Aryabhata for evaluating sine function,
  - the summation of series & and finding the sum of sums, the cow problem, etc. as well as
  - the method of arriving at infinite series and its fast convergent approximations,

  indicate that Indian approach to mathematics has been more algorithmic in nature and also practical/application oriented.

[4] This was 3 decades ago, but the situation is not 'very' different today.

Concluding Remarks

- The texts on Indian mathematics, soon after enunciating a rule or principle present plenty of examples drawn from day to day life, that are at once interesting, as well as drive in the principles.
- Both the rules and examples are presented in the form of *sūtras* and verses that can be easily committed to memory.
- The act of memorizing besides generating fun could also be of help in developing certain 'desirable' mental faculty.
- Studying history would help us get away with the wrong notions generated by Euro-centric bias or Indi-centric bias.
- Finally, making the students aware of the major achievements of their own composite culture—particularly in their impressionable age—is likely to boost their self-confidence and self-esteem which are important ingredients in fostering creativity.

Thanks!

॥ धन्यवादाः ॥