History of Indian Mathematics and its Implications for Mathematical Education

with focus on

Bhadragaṇita of *Nārāyaṇa*

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Introduction

- A few years ago, an unusual seminar series was held at Chennai Mathematical Institute (CMI) and the Proceedings that was brought out.
- In his scholarly preface, David Mumford observes:[1]

  > *Too many people still think that mathematics was born in Greece and more or less slumbered until the Renaissance.*

- The state described by Mumford is primarily due to the fact that the knowledge remained confined among the specialists.
- Consequently, the general picture that emerged from the books on history of mathematics with regard to the Indian contributions till recent times were quite incomplete and at times misleading.

[1] *Contributions to the History of Indian Mathematics*, Ed. C. S. Seshadri, Hindustan Book Agency, Delhi 2010.

Introduction

Morris Kline[2] observes:

- Sometimes the Hindus were aware that a formula was only approximately correct and sometimes they were not. Their values of π were generally inaccurate; . . .
- In trigonometry the Hindus made a few minor advances[3] . . .
- As our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the deductive patterns.
- There is much good procedure and technical facility, but no evidence that they considered proof at all.[4]

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[2]Credited with more than a dozen books on various aspects of mathematics such as history, philosophy, and pedagogy.

[3]This observation is despite the fact that Hindus have extensively employed *Jīveparasparanyāya*, and obtained complicated results. . .

[4]Morris Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York 1972, pp. 188-190.

The *Sulbasūtras*

- The word *śulba* stems from the root *śulb* (to measure).
- Since all the measurements were done using ropes or chords, in the very early times, it seems the word in due course was synonymously employed to refer to the chords themselves.
- The etymological derivation of the word *śulba* (referring to a rope) can be given as:

  शुल्बयत्यनेन इति शुल्बः ।[5]

- The exact derivation of the compound word *Śulbasūtras*, including the grammatical peculiarities is:

  शुल्बनम् = शुल्बः (शुल्ब् + घञ्)[6] । तत्सम्बन्धि सूत्राणि ।

- Seven *Śulbasūtras*, namely *Baudhāyana, Āpastamba, Kātyāyana, Mānava, Maitrāyaṇa, Vāraha* and *Vādhūla* are extant today.

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[5] This derivation is technically referred to as *karaṇavyutpatti*.

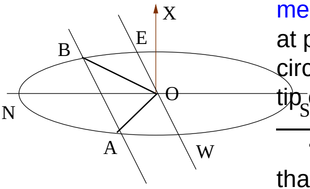
[6] This type of derivation is known as *bhāvavyutpatti* and the *sūtra* that comes into play is '*bhāve ghañ*'.

Dertermining the east-west line

- Determining the exact east-west line at a given location, is a pre-requisite for all constructions, be it a residence, a temple, a sacrificial altar or a fire-place.
- The procedure for its determination is described thus:

समे शङ्कुं निखाय ‘शङ्कुसम्मितया रज्ज्वा’ मण्डलं परिलिख्य यत्र लेखयोः शङ्क्वग्रच्छाया निपतति तत्र शङ्कू निहन्ति सा प्राची। [*Kt. Su.* I 2]

OA= forenoon shadow
OB= afternoon shadow

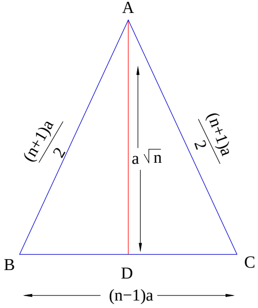


Fixing a pin (or gnomon) on level ground and drawing a circle with a cord measured by the gnomon,[a] he fixes pins at points on the line (of the circumference) where the shadow of the tip of the gnomon falls. That is the $pr\bar{a}c\bar{\imath}$.

[a] For the cord to be measured by gnomon, its length r
that of the gnomon. This prescription has important astr

To construct a square that is *n* times a given square

- Kātyāyana gives an ingenious method to construct a square whose area is *n* times the area of a given square.



यावत्प्रमाणानि समचतुरश्राणि एकीकर्तुं
चिकीर्षेत् एकोनानि तानि भवन्ति तिर्यक्।
द्विगुणान्येकत एकाधिकानि त्र्यस्रिर्भवति।
तस्येषुः तत्करोति। [*Kt.Sl* VI 7]

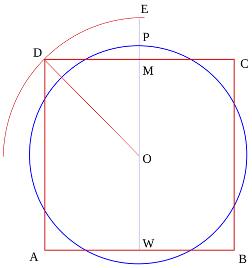
As much . . . one less than that forms the base . . . the arrow of that [triangle] makes that (gives the required number $\sqrt{n}$).

- In the Figure $BD = \frac{1}{2}BC = (\frac{n-1}{2})a$. Consider $\triangle ABD$,

$$AD^2 = AB^2 - BD^2 = \left(\frac{n+1}{2}\right)a^2 - \left(\frac{n-1}{2}\right)a^2$$

$$= \frac{a^2}{4}\ (n+1)^2 - (n-1)^2 = \frac{a^2}{4} \times 4n = (na^2)$$

To transform a square into a circle



According to Baudhāyana:

चतुरश्रं मण्डलं चिकीर्षन् अक्ष्णयार्धं मध्यात् प्राचीम् अभ्यपातयेत् यद्यदतिशिष्यते तस्य सह तृतीयेन मण्डलं परिलिखेत्।

अक्ष्णयार्धं = semi-diagonal
मध्यात् प्राचीम् = from centre to the east
यद्यदतिशिष्यते = whatever [portion] remains
तस्य सह तृतीयेन = with one-third of that

$$\begin{aligned} AB &= 2a \\ OP &= r \\ OD &= a\sqrt{2} \\ ME &= a(\sqrt{2}-1) \end{aligned}$$

$$\text{The radius } OP = r = a + \frac{a}{3}(\sqrt{2}-1) = \frac{a}{3}(2+\sqrt{2}).$$

How to find $\sqrt{2}$?

To transform a square into a circle

How did the *Sulvakāras* specify the value of $\sqrt{2}$?

- The following *sūtra* gives an approximation to $\sqrt{2}$:

  प्रमाणं तृतीयेन वर्धयेत्, तच्चतुर्थेन, आत्मचतुस्त्रिंशोनेन,
  सविशेषः।

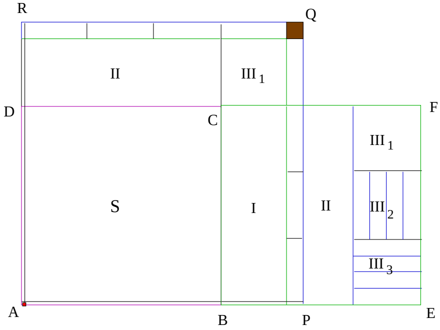
$$
\begin{aligned}
\sqrt{2} &= 1+\frac{1}{3}+\frac{1}{3\times 4}-\frac{1}{3\times 4\times 34} \qquad (1)\\
&= \frac{577}{408}\\
&= 1.4142156\ldots
\end{aligned}
$$

- How did the *Śulvakāras* arrive at the above expression?
- Several explanations have been offered over the last centuries, of which one of them is geometrical construction.

Approximation for $\sqrt{2}$

Rationale for the expression $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$ by Geometrical Construction

- Consider two squares *ABCD* and *BEFC* (sides of unit length).
- The second square *BEFC* is divided into three strips.
- The third strip is further divided into many parts, and these parts are rearranged (as shown) with a void at *Q*.
- Now, each side of the new square $APQR = 1 + \frac{1}{3} + \frac{1}{3.4}$.



Approximation for $\sqrt{2}$

Rationale for the expression (contd.)

- The area of the void at Q is $\left(\frac{1}{3.4}\right)^2$.
- Suppose we were to strip off a segment whose breadth is x from either side of this square, such that the area of the stripped off portion is exactly equal to that of the void at Q, then we have,

$$2x\left(1+\frac{1}{3}+\frac{1}{3.4}\right)-x^2=\left(\frac{1}{3.4}\right)^2.$$

- Neglecting x^2 (as it is too small), we get

$$x=\left(\frac{1}{3.4}\right)^2\times\frac{3.4}{34}=\frac{1}{3.4.34}.$$

- Hence the side of the resulting square

$$\sqrt{2}=1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}$$

Saṅkalita & *Vārasaṅkalita*

Sum of series & Sum of sums

- The results

$$\sum_{r=1}^{n} r = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

and $$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

were known to the authors of AB, GSS, PG, L.

- Now we consider sum of sums, for which we use the notation,

$$V_n^{(1)} = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$V_n^{(2)} = V_1^{(1)} + V_2^{(1)} + \ldots V_n^{(1)}$$

$$= \sum_{r=1}^{n} \frac{r(r+1)}{2} = \frac{n(n+1)(n+2)}{1.2.3}$$

Vārasaṅkalita

k^{th} order repeated sum of natural numbers (in *Gaṇitakaumudī*)

Nārayaṇa presents the formula in the following *Āryā*:

- एकाधिकवारमिताः पदादिरूपोत्तराः पृथक् तेऽभ्यस्ताः।
  एकाद्येकचयहृताः तद्घातो वारसङ्कलितम्॥
  - पदादिरूपोत्तराः – $n, \; n+1, \; n+2, \ldots$
  - एकाधिकवारमिताः – limited to $k+1$ terms
  - एकाद्येकचयहृताः – 1, 2, 3, ... are the divisors
  - तद्घातो वारसङ्कलितम् – the product is *vārasaṅkalita*
- The sum of sums

$$V_n^{(k)} = V_1^{(k-1)} + V_2^{(k-1)} + \ldots V_n^{(k-1)}$$

is stated to be

$$V_n^{(k)} = \frac{n(n+1)\ldots(n+k)}{1.2.3.\ldots k+1} = {}^{n+k}C_{k+1}$$

The Cow Problem

प्रतिवर्षं गौः सूते वर्षत्रितयेन तर्णकी तस्याः।
विद्वन् विंशतिवर्षैः गौरेकस्याश्च सन्ततिं कथय॥

A cow gives birth to a [she] calf every year [and] their calves themselves [begin giving birth], in 3 years time. O learned, tell the number of progeny produced by a cow in 20 years.

Recalling

$$V_n^{(0)} = 1 + 1 + \ldots + 1 = n$$

$$V_n^{(1)} = V_1^{(0)} + \ldots + V_n^{(0)} = 1 + 2 + \ldots + = \frac{n(n+1)}{2}$$

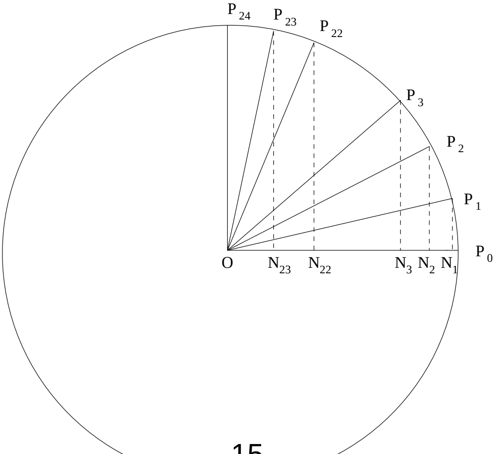
$$V_n^{(2)} = V_1^{(1)} + V_2^{(1)} + \ldots + V_n^{(1)} = \frac{n(n+1)(n+2)}{1.2.3}$$

The Cow Problem

Year	1^{st} gen.	2^{nd} gen.	3^{rd} gen.	4^{th} gen.	5^{th} gen.	6^{th} gen.	7^{th} gen.
1	1						
2	1						
3	1						
4	1	$V_1^{(0)}$					
5	1	$V_2^{(0)}$					
6	1	$V_3^{(0)}$					
7	1	$V_4^{(0)}$	$V_1^{(1)}$				
8	1	$V_5^{(0)}$	$V_2^{(1)}$				
9	1	$V_6^{(0)}$	$V_3^{(1)}$				
10	1	$V_7^{(0)}$	$V_4^{(1)}$	$V_1^{(2)}$			
11	1	$V_8^{(0)}$	$V_5^{(1)}$	$V_2^{(2)}$			
12	1	$V_9^{(0)}$	$V_6^{(1)}$	$V_3^{(2)}$			
13	1	$V_{10}^{(0)}$	$V_7^{(1)}$	$V_4^{(2)}$	$V_1^{(3)}$		
14	1	$V_{11}^{(0)}$	$V_8^{(1)}$	$V_5^{(2)}$	$V_2^{(3)}$		
15	1	$V_{12}^{(0)}$	$V_9^{(1)}$	$V_6^{(2)}$	$V_3^{(3)}$		
16	1	$V_{13}^{(0)}$	$V_{10}^{(1)}$	$V_7^{(2)}$	$V_4^{(3)}$	$V_1^{(4)}$	
17	1	$V_{14}^{(0)}$	$V_{11}^{(1)}$	$V_8^{(2)}$	$V_5^{(3)}$	$V_2^{(4)}$	
18	1	$V_{15}^{(0)}$	$V_{12}^{(1)}$	$V_9^{(2)}$	$V_6^{(3)}$	$V_3^{(4)}$	
19	1	$V_{16}^{(0)}$	$V_{13}^{(1)}$	$V_{10}^{(2)}$	$V_7^{(3)}$	$V_4^{(4)}$	$V_1^{(5)}$
20	1	$V_{17}^{(0)}$	$V_{14}^{(1)}$	$V_{11}^{(2)}$	$V_8^{(3)}$	$V_5^{(4)}$	$V_2^{(5)}$
Sum	20	153	560	1001	762	210	8

Construction of the sine-table (Āryabhaṭa's method)

- A quadrant is divided into 24 equal parts, so that each arc bit $\alpha = \frac{90}{24} = 3^\circ 45' = 225'$.
- Procedure for finding $R \sin i\alpha, (P_i N_i)$ $i = 1, 2, \ldots 24$ is given.
- The R sines of the intermediate angles are determined by interpolation (I order or II order).



Recursion relation for the construction of sine-table

$\bar{A}ryabhaṭ\bar{\imath}ya$'s algorithm for constructing of sine-table

- The content of the verse in $\bar{A}ryabhaṭ\bar{\imath}ya$ translates to:

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin i\alpha - R\sin(i-1)\alpha - \frac{R\sin i\alpha}{R\sin\alpha}.$$

- In fact, the values of the 24 *Rsines* themselves are explicitly noted in another verse.
- The exact recursion relation for the Rsine differences is:

$$R\sin(i+1)\alpha - R\sin i\alpha = R\sin i\alpha - R\sin(i-1)\alpha - R\sin i\alpha \;\; 2(1-\cos\alpha).$$

- Approximation used by Āryabhaṭa is $2(1-\cos\alpha) = \frac{1}{225}$.
- While, $2(1-\cos\alpha) = 0.0042822$, $\frac{1}{225} = 0.00444444$.
- In the recursion relation provided by Nīlakaṇṭha we find $\frac{1}{225} \rightarrow \frac{1}{233.5} (= 0.0042827)$.

Comment on Āryabhaṭa's Method (Delambre)

Commenting upon the method of Āryabhaṭa in his monumental work Delambre[7] observes:

> "*The method is curious: it indicates a method of calculating the table of sines by means of their second differences. . . . The differential process has not up to now been employed except by Briggs, who himself did not know that the constant factor was the square of the chord . . . Here then is a method which the Indians possessed and which is found neither amongst the Greeks nor amongst the Arabs.*"[8]

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[7] ". . . an astronomer of wisdom and fortitude, able to review 130 years of astronomical observations, assess their inadequacies, and extract their value." – Prix prize citation 1789.

[8] Delambre, *Historie de l'Astronomie Ancienne*, t 1, Paris 1817, p.457; cited from B. Datta and A. N. Singh, Hindu Trigonometry, IJHS 18, 1983, p.77.

Magic Squares

Background and Relevance

- Today there is a lot of discussion going on all around the world to see how to make mathematics learning more interesting. 1000s are working on what is called recreational mathematics.
- As far as Arithmetics is concerned, certainly one of the ways to make it interesting is to introduce the topic of Magic Squares—called *Bhadra-gaṇita* in Indian Mathematics.
- The nomenclature stems from the fact it was considered to fetch ***bhadra***—an all round well-being—just like *yantras*, wherein we have various letters inscribed.
- The earliest extant mathematical text in India that presents some detailed treatment on the topic is *Gaṇitasārakaumudi* of Thakkura Pherū (c. 1300 CE).
- A more detailed mathematical treatment, by way of exclusively devoting a chapter (chap. 14, consisting of 75+ verses), is provided by Nārāyaṇa in his *Gaṇitakaumudī* (c. 1356).

Normal and Pan-diagonal Magic squares

- Depending on the number of variant ways in which one can get the desired sum, magic squares have been classified into:
  - semi-magic (only rows and columns sum up to the no.)
  - magic, and (rows, columns & principal diagonals)
  - pan-diagonal magic (the above, plus the broken diagonals)
- An example of a magic square and a pan-diagonal (PD) magic square.

A normal Magic Square

(Sum = 34)

12	3	6	13
14	5	4	11
7	16	9	2
1	10	15	8

PD Sum: $3 + 4 + 2 + 1 \neq 34$

A Pan-diagonal Magic Square

(Sum = 34)

2	8	17	13
16	14	1	9
3	7	18	12
19	11	4	6

PD Sum: $8 + 1 + 12 + 19 = 40$

Classification of Magic Squares

- Thakkura Pherū in his *Gaṇitasārakaumudī* classifies $n \times n$ magic squares into the following types:
  - *Samagarbha* (n doubly-even or of the form $4m$)
  - *Viṣamagarbha* (n singly-even or of the form $4m + 2$)
  - *Viṣama* (n is odd)
- Having made this classification, Pherū presents a few examples of magic squares—that are non pan-diagonal.
- Moreover, they are "*normal*" magic squares of order $n = 3, 4, 5, 6, \ldots,$ whose magic sum are $S = 15, 34, 65, 111, \ldots$.
- In these squares, the entries in the n^2 cells will be sequence of natural numbers $1, 2, \ldots n^2$ and the magic sum will be $\boxed{S = \frac{n(n^2+1)}{2}.}$
- However, in the pan-diagonal magic square described by Nārāyaṇa the sum S need not be magic sum given above.

Purpose and classification (according to Nārāyaṇa)

- The purpose of magic squares has been delineated thus:

  सद्गणितचमत्कृतये यन्त्रविदां प्रीतये कुगणकानाम्।
  गर्वक्षित्यै वक्ष्ये तत्सारं भद्रगणिताख्यम्॥[9]

- Classifying the magic squares Narayana observes:

  समगर्भविषमगर्भे विषमञ्चेति त्रिधा भवेद् भद्रम्।

- Defines them as follows:

  भद्राङ्के चतुराप्ते निरग्रके तद्भवेच्च समगर्भम्।
  द्व्यग्रे तु विषमगर्भं त्र्येकाग्रे केवलं विषमम्॥

  When the order of the magic square is divided by 4, if the remainder $r = 0$, then it is samagarbha; if $r = 2$, then it is viṣamagarbha; and if $r = 3$ or 1, then it is viṣama.

[9] *Gaṇitakaumudī* 14.2

Preliminaries to *Bhadragaṇita* presented by Nārāyaṇa

- One of the notable features of Nārāyaṇa is that he methodically introduces all topics that he discusses (unlike some of his predecessors like Āryabhaṭa). For instance, he sets apart 5 verses to introduce the preliminaries to *Bhadragaṇita*

सर्वेषां भद्राणां श्रेढीरीत्या भवेद् गणितम्।
येषां गणितमभीष्टं साध्यौ तेषां मुखप्रचयौ। ...
यद्यावन्ति गृहाणि श्रेढीविषये भवेद् गच्छः।
भद्रे कृतिगतकोष्ठे तन्मूलं जायते चरणः।
इह नारायणविहिता परिभाषा भद्रगणिते च॥

- In all magic squares arithmetical progression will be involved.
- The first term and the common difference.
- The number of boxes in the square will be equal to the number of terms in the arithmetical sequence.

Sum of the sequence and the Magic Sum

- The formula that is applicable in the case of the ‘normal’ magic squares is set forth by $N\bar{a}r\bar{a}ya\d{n}a$ in the following verse.

  सपदः पदवर्गोऽर्धं रूपादिचयेन भवति सङ्कलितम्।
  तत् पदमूलेन हृतं फलं भवेदिष्टभद्रे वै॥

- The first half of the above verse translates to

$$Sankalita = \frac{pada^2 + pada}{2}, \tag{2}$$

- And the second half to

$$Bhadraphalam = \frac{Sankalita}{\sqrt{pada}}. \tag{3}$$

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Sampuṭavidhi* given by Nārāyaṇa

समगर्भे द्वे कार्ये छादकसंज्ञं तयोर्भवेदेकम्।
छाद्याभिधानमन्यत् करसम्पुटवच्च सम्पुटो ज्ञेयः॥
इष्टादीष्टचयाङ्का भद्रमिता मूलपङ्क्तिसंज्ञाद्या।

- Consider two $n \times n$ squares where $n = 4m$.
- Of the two, one is called the coverer (*chādaka*)
- The other is called the covered (*chādya*)
- The folding here is just like folding the palms.
- The first sequence known as *mūlapaṅkti* [has],
- any desired number as the first term (*iṣṭādi*) and so too the common difference (*caya*) [and]
- is limited by the order of the magic square

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Sampuṭavidhi* given by Nārāyaṇa

तद्वदभीप्सितमुखचयपङ्क्तिश्च[10] अन्या पराख्या स्यात्॥

- Similarly, (another) sequence having desired number as the first item and also as the common difference is known as the *parāpaṅti*.

Given below are a few examples of *mūlapaṅti* and *parāpaṅti*

mūlapaṅti				*caya*	*parāpaṅti*				*caya*
1	2	3	4	1	0	1	2	3	1
2	4	6	8	2	1	2	3	4	1
3	6	9	12	3	2	3	4	5	1
3	6	9	12	3	4	6	8	10	2
4	8	12	16	4	0	3	6	9	3

[10] The vigraha is: मुखञ्च चयश्च मुखचयौ। अभीप्सितौ मुखचयौ अभीष्टमुखचयौ। तौ यस्याः पङ्क्तेः सा अभीप्सितमुखचयपङ्क्तिः।

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Sampuṭavidhi* given by Nārāyaṇa

मूलाख्यपङ्क्तियोगोनितं फलं परसमाससंभक्तम्।
लब्धहतापरपङ्क्तिः गुणजाख्या सा भवेत् पङ्क्तिः॥

- The result obtained by decreasing the sum of the *mūlapaṅti* [from the desired magic sum],
- when divided by the sum of the *parapaṅti* [is the *guṇa*].
- The elements of the *parapaṅti* multiplied by that *guṇa* obtained is known as the *guṇapaṅti*.

Example 1: Suppose the desired sum $S = 40$

- *Mūla-paṅti* – | 1 | 2 | 3 | 4 |. Its sum $s_m = 10$
- *Parā-paṅti* – | 0 | 1 | 2 | 3 |. Its sum $s_p = 6$. Now,

$$\frac{S - s_m}{s_p} = \frac{40 - 10}{6} = 5$$

- Using this we obtain *Guṇa-paṅti* – | 0 | 5 | 10 | 15 |.

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Samputavidhi* given by Nārāyaṇa

मूलगुणाख्ये पङ्क्ती ये ते भद्रार्धतस्तु परिवृत्ते।

- The two sequences that are known as *mūlapaṅti* and *guṇapaṅti*,
- are to be arranged [in the cells] in a cyclic manner—clockwise or anti-clockwise[11]—from the centre of the magic square.

- *Mūla-paṅti* – | 1 2 3 4 |.
- *Guṇa-paṅti* – | 0 5 10 15 |.

clockwise	2	3	2	3
	1	4	1	4
anti-clockwise	3	2	3	2
	4	1	4	1

clockwise		anti-clockwise	
10	15	5	0
5	0	10	15
10	15	5	0
5	0	10	15

[11] specified so in the next few lines of the verses.

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Sampuṭavidhi* given by Nārāyaṇa

ऊर्ध्वस्थितैः तदङ्कैः छादकसञ्छाद्ययोः पृथग्यानि॥
तिर्यक्कोष्ठान्याद्ये अन्यतरस्मिन्नूर्ध्वगानि कोष्ठानि।
भद्रस्यार्धं क्रमगैः उत्क्रमगैः पूरयेदर्धम्।

- The horizontal blocks in the first and the vertical ones in the other
- Half of the magic square *bhadra* is to be filled in order
- And the other half in the reverse order

2	3	2	3	clockwise
1	4	1	4	
3	2	3	2	anti-clockwise
4	1	4	1	

clockwise		anti-clockwise	
10	15	5	0
5	0	10	15
10	15	5	0
5	0	10	15

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Sampuṭavidhi* given by Nārāyaṇa

भद्राणामिह सम्पुटविधिः उक्तो नृहरितनयेन॥

- Here the folding method of constructing magic squares, has [thus] been explained by the son of Nṛhari (Nārāyaṇa Paṇḍita).

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

छाद्य-square

+ (folded)

10	15	5	0
5	0	10	15
10	15	5	0
5	0	10	15

छादक-square

=

2	8	17	13
16	14	1	9
3	7	18	12
19	11	4	6

भद्र-square

Notable features:

- Apart from the rows, columns and the principal diagonals, the broken diagonals too add up to the magic sum (pan-diagonal).
- The 15 distinct quadruplets that can be considered also add up to the magic sum.

A few examples of 4 $\times$ 4 Magic Squares

Example 2: Suppose the desired sum $\boxed{S = 120}$

- *Mūla-paṅti* – | 2 | 4 | 6 | 8 |. Its sum $s_m = 20$
- *Parā-paṅti* – | 1 | 2 | 3 | 4 |. Its sum $s_p = 10$. Now,

$$\frac{S - s_m}{s_p} = \frac{120 - 20}{10} = 10$$

- Using this we obtain *Guṇa-paṅti* – | 10 | 20 | 30 | 40 |.

4	6	4	6
2	8	2	8
6	4	6	4
8	2	8	2

छाद्य-square

$+$ (folded)

30	40	20	10
20	10	30	40
30	40	20	10
20	10	30	40

छादक-square

=

14	26	44	36
42	38	12	28
16	24	46	34
48	32	18	22

भद्र-square

A few examples of 4 × 4 Magic Squares

Example 3: Suppose the desired sum $\boxed{S = 120}$. Though the magic sum is the same as the previous example, here we choose completely different *mūla* and *parā-paṅti*.

- *Mūla-paṅti* – $\boxed{3 \quad 6 \quad 9 \quad 12}$. Its sum $s_m = 30$
- *Parā-paṅti* – $\boxed{0 \quad 3 \quad 6 \quad 9}$. Its sum $s_p = 18$. Now,

$$\frac{S - s_m}{s_p} = \frac{120 - 30}{18} = 5$$

- Using this we obtain *Guṇa-paṅti* – $\boxed{0 \quad 15 \quad 30 \quad 45}$.

6	9	6	9
3	12	3	12
9	6	9	6
12	3	12	3

छाद्य-square

+ (folded)

30	45	15	0
15	0	30	45
30	45	15	0
15	0	13	45

छादक-square

=

6	24	51	39
48	42	3	27
9	21	54	36
57	33	12	18

भद्र-square

A few examples of 4 × 4 Magic Squares

Example 4: Suppose the desired sum $\boxed{S = 128}$. The examples choosen so far just happened to be multiples of 10. This example is choses to show that this need not be so.

- *Mūla-paṅti* – 3 6 9 12. Its sum $s_m = 30$
- *Parā-paṅti* – 2 3 4 5. Its sum $s_p = 14$. Now,

$$\frac{S - s_m}{s_p} = \frac{128 - 30}{14} = 7$$

- Using this we obtain *Guṇa-paṅti* – 14 21 28 35.

6	9	6	9
3	12	3	12
9	6	9	6
12	3	12	3

छाद्य-square

+ (folded)

28	35	21	14
21	14	28	35
28	35	21	14
21	14	28	35

छादक-square

=

20	30	41	37
38	40	17	33
23	27	44	34
47	31	26	24

भद्र-square

The *Śulva* theorem (***Bhujā-koṭi-karṇa-nyāya***)

- A clear enunciation of the so-called 'Pythagorean' theorem is found in the *Śulva-sūtras*:[12]

  दीर्घचतुरश्रस्य अक्ष्णयारज्जुः[13] पार्श्वमानी तिर्यङ्मानी च यत् पृथग्भूते कुरुतः तदुभयं करोति।

  The square of the diagonal of a rectangle is the sum of the squares of the lateral and the vertical sides.

- By now, this *sūtra* is fairly well known.
- The purpose of bringing up this topic is to present the interesting discussion and the dissectional proof presented by Nīlakaṇṭha in his *Siddhāntadarpaṇa*.

[12] *Bodhāyana Śulvasūtra* 1.48.

[13] The word अक्ष्णया is an indeclinable. It occurs in the Vedic literature at several places: अक्ष्णया व्याघारयति।...तस्मादक्ष्णया पशवोङ्गानि प्रतितिष्ठन्ति।

Bhujā-koṭi-karṇa-nyāya

Two possible ways of demonstration

1. सङ्ख्यान्वेषणम् : Demonstration by enumerating

   द्वयो राश्योर्वर्गयोगमूलं निरवयवम् एकादिसङ्ख्याविशेषेषु कयोः कयोः स्यादिति सङ्ख्या निश्चीयते। तदन्वेषण एको मार्गः।

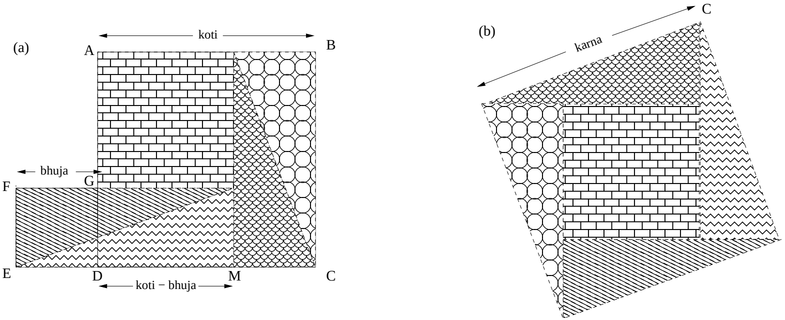
2. क्षेत्रच्छेदनम् : Demonstration by area dissection method

   अशेषविषयव्यापी क्षेत्रच्छेदकद्वारः अन्यो मार्गः।

Choosing the second over the first, Nīlakaṇṭha states the reason for his choice:

- अल्पविषयत्वात् : limited
- आनन्त्यात् : the set will be enormous in size
- अविश्रान्तेः : and will be never ending.

Kṣetra-chedana: Area dissectional proof



एतत्सर्वं युक्तिमूलमेव,न त्वागममूलम्।

All this is based on reasoning alone and does not need the authority of any authentic text [for it validity].

The number of terms in an arithmetical series

- Consider an arithmetical series of the form –

$$a + (a + d) + (a + 2d) + (a + 3d) \ldots\ldots + (a + (n - 1)d). \quad (4)$$

- The formula for finding the number terms n in the series, in terms of it sum S, the first term a and the common difference d is encoded in the following verse:[14]

गच्छोऽष्टोत्तरगुणिताद् द्विगुणाद्युत्तरविशेषवर्गयुतात्।
मूलं द्विगुणाद्यूनं स्वोत्तरभाजितं सरूपार्धम् ॥

- The content of the above verse can be expressed as:

$$n = \frac{1}{2}\left(\frac{\sqrt{8Sd + (2a - d)^2} - 2a}{d} + 1\right) \quad (5)$$

[14] Āryabhaṭa, *Āryabhaṭīya*, *Gaṇitapāda*, verse 20.

Approximations to π

- The commentary *Kriyākramakarī* while presenting the ratio of the circumference to the diameter given by different *Ācāryas* observes:.

माधवाचार्यः पुनः अतोप्यासन्नतमां परिधिसङ्ख्यामुक्तवान् –
विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः ॥[15]

- The values of π given in the above verses is:

$$\pi = \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592 \quad \text{(correct to 11 places)}$$

[15] *Vibudha*=33, *Netra*=2, *Gaja*=8, *Ahi*=8, *Hutāśana*=3, *Triguṇa*=3, *Veda*=4, *Bha*=27, *Vāraṇa*=8, *Bāhu*=2, *Nava-nikharva*=9×10^{11}. (The word *nikharva* represents 10^{11}).

Concluding Remarks

- Introducing a bit of history would make the mathematics education more complete.
- It would enable us to have a multi-cultural perspective which is presently lacking in the educational curricula.
- The Indian approach to a problem being algorithmic, it would enable the students to have a sense of a different flavor of mathematics. For example,
  - the prescription in *Śulbasūtra* texts for finding surds,
  - the recurrence relation given by Aryabhata for evaluating sine function,
  - the techniques for solving indeterminate equations given by Brahmagupta and Bhaskara,
  - the method of arriving at infinite series
  - finding fast convergent approximations to the 'Gregory-Leibniz' series for π

Concluding Remarks

- Some of the unique techniques along with the illustrative examples and demonstrations provided in the texts, could perhaps prove to be simpler for the students in assimilating the concepts with much ease.
- There could be fun in memorizing the formula (for instance, for the number of terms in an arithmetical series) in the form of verses.
- Knowing history would help us in getting away with the false pictures that have already been created by some of the accounts—be it generated by Euro-centric bias or Indi-centric bias.
- Lastly, making the students aware of the major achievements of their own ancestors—particularly in their impressionable age—is quite likely to boost their self-confidence and also provide the necessary motivation in building a self-reliant nation.

Thanks!

॥ धन्यवादः ॥