Transitions: school mathematics and the discipline of mathematics

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- I asked the class to argue this: as long as water supply was regular, lorries must come on all days of the week, eventually.
- We had a nice discussion and the class was indeed convinced of this.

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- After several trials, the children get a characterization.
- They draw some beautiful pictures.

The game has led to an interesting exercise on *finite groups* at undergraduate class in CMI, Chennai.

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- Perhaps not surprisingly, these transitions are co-located with what are considered *difficult* topics for teaching / learning.
- They sharply contrast school mathematics and the discipline of mathematics.
- Understanding these also offers hope for solutions to the difficulties mentioned above.

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- From the assumed infinite to the explicit finite. (Difficulty with problems in combinatorics and numer theory.)

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- Why does there exist a real number as solution to $x^2 = 2$?
- What is π^2 ?
- It seems reasonable that dividing a ribbon of length ℓ among three persons should get $\ell/3$ units for each. But what about cutting it up into pieces $3cm$ long generating $\ell/3$ pieces ?

More examples

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- Geometrical reasoning: The difficulty of not knowing what to look for.
- Probabilistic reasoning: abstraction and imagination well beyond observable phenomena.
- Mathematical modelling: Part of curricula but the modelling never challenges the student's mathematical conceptalization.

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The last is a transition that has nothing to do with developmental stages, and is the big difference between school mathematics (as is) and the discipline.

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- University teachers routinely complain about the quality of student preparation at the entry level.
- Competence in mathematics is often equated with success in competitive examinations.
- One main reason for all this: the big gap between mathematics in school classrooms and that at UG level and contributions to the *knowledge economy*.

The practice of mathematics

Doing mathematics often means:

- Selecting between representations or devising new ones,
- Looking for invariances,
- Observing extreme cases and typical ones to come up with conjectures,
- Looking actively for counterexamples,
- Simplifying or generalizing problems to make them easier to address,
- Building on answers to generate new questions for exploration,

and so on. These are mostly missing at school.

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- Curriculum needs to be shaped so that processes such as estimation, approximation, visualization, representation, reasoning, argumentation, making connections, etc can be emphasized in the classroom.
- This is within the realm of feasibility, but requires considerable re-orientation of classrooms, textbooks and systemic expectations.

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- This can also be a way of ensuring that every child's attention is engaged, and remove the sense of fear and failure that many children experience with regard to mathematics.
- This means that we need to de-emphasize the tall and sequential nature of mathematics at school.

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- Limited time slots, curricular pressures, and assessment regimes.
- The role of educational authority.

Teachers

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- In Felix Klein's words, mathematics teachers suffer due to double discontinuity.
- Many teachers had themselves not negotiated the transitions successfully and lack introspection on these difficulties.
- The advocated shift requires knowledgeable teachers, but most teachers do not have personal experience of what it means to do mathematics over time, exploring questions which have intellectual purpose, not pedagogic purpose.

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- Analysts who (convincingly) point to impact of socio-cultural factors on classrooms and learners ?

They all seem to be interesting, and in the end, inadequate.

National frameworks

Approaches to this can perhaps be found in articulated differences in National curricula. As an example, relative weightage of topics in the Board exams in Hungary:

Item	Topic	% weight
a.	Sets, logics, combinatorics, graphs	25
b.	Relations, functions, the elements of calculus	20
c.	Arithmetic, algebra, number theory	20
d.	Probability, statistics	15
e.	Geometry, coordinate geometry, trigonometry	20

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- Israel is a rare instance of expanding and increasing mathematics content.
- The RME approach in the Netherlands was designed to build mathematics learning from relevant problem contexts.

A quote

Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now. . . .
The pupil himself should reinvent mathematics. During this process, the learner is engaged in an activity where experience is described, organized and interpreted by mathematical means. This activity is mathematising.

Hans Freudenthal, Revisiting Mathematics Education, 1991.

Discussion time

Thank you.

Questions, comments, suggestions welcome; also, please write to jam@imsc.res.in.