

HOW RESEARCH MATHEMATICIANS MAKE SENSE OF MATHEMATICAL DEFINITIONS

REVATHY PARAMESWARAN

P. S. SENIOR SECONDARY SCHOOL

33 ALARMELUMANGAPURAM

MYLAPORE, CHENNAI 600004

INDIA

E-MAIL:REVATHY@CMI.AC.IN

In this paper we report on our study of the cognitive tools that research mathematicians employ when they comprehend abstract mathematical definitions. We find that having a supply of test examples is essential to understanding abstract definitions at all levels of mathematical knowledge acquisition. Our study also confirms that having and resolving conflicts is an important factor in obtaining a deeper understanding of definitions.

Keywords: Definitions, examples, conflicts, understanding.

INTRODUCTION

It is common knowledge that definitions play a pivotal role in mathematics. Research into how students make sense of mathematical definitions reveals that learners encounter different types of obstacles, both epistemological and cognitive. According to Vinner(1991), serious difficulties in comprehending definitions can be attributed to the dichotomy that exists between the structure of mathematics as conceived by professional mathematicians and the cognitive processes involved in concept acquisition. Hence it is instructive to understand how mathematicians construct definitions and the cognitive processes they use when they attempt to understand definitions created by their peers.

With this goal in mind, we prepared a questionnaire and sent it to some research mathematicians from different parts of the world. Their responses provided us with an insight into the different strategies that mathematicians employ while comprehending new definitions.

THEORETICAL PERSPECTIVE AND LITERATURE

We find the theoretical framework of Pirie and Kieren (1994) relevant to our study. Their theory describes growth of 'understanding' over a period of time. The essence of their theory is that understanding is not always continuous and learners often retreat to their previous ways of thinking,

emerging forward with more sophisticated and deeper understanding. They use an onion-layer model to depict different levels of understanding. The process of coming to know a mathematical object is what they call as *primitive knowing*. The first recursion occurs when the learner forms images out of these effective actions. When these images take a definite form, the level is called *image having*. This is the first level of abstraction, as the learner does not need to take actions as examples and the abstraction is constructed by the learner and not imposed from outside. These images can now be examined for properties, distinctions and so on. This level is called *property noticing*. When this is done at a conscious level and when one discards the origin of one's own action the level is called *formalizing*. It is at this level the mathematical objects become defined for the learner and begin to exist as an independent entity. At the next level he tries to achieve consistency in his thought processes by trying to accommodate the newly acquired knowledge with the existing structure. This level is called *observing*. When he is able to place his thought processes into an axiomatic structure the level is called *structuring*. At the next level he is able to freely create new mathematical structures with the previous knowledge structures acting as the initiating ground. At this level which is the highest level of recursion, the learner begins to function independently. This level is rightly called *inventing*.

It is very important to note that the levels do not correspond to levels in mathematics but in understanding. Thus this theory reflects understanding as a personal knowledge construction process since construction of knowledge takes place by folding back to previous experiences which colours the present structure. These experiences are experiential and individualistic.

Vinner and Tall (1981) have provided a framework for understanding how one comprehends and uses a mathematical definition. According to Vinner and Tall, to each mathematical concept, a *concept definition* and a *concept image* are associated.

Concept image is the total cognitive structure associated with the mathematical concept in the individual's mind. The form of words that is used to describe the concept image is called the *concept definition*. This could be formal and given to the individual as a part of a formal theory or it may be a personal definition invented by an individual describing his concept image. A *potential conflict factor* is any part of the concept image which conflicts with another part or any implication of the concept definition. A *cognitive conflict* is created when two mutually conflicting factors are evoked simultaneously in the mind of an individual. The potential conflict may not become a cognitive conflict if the implications of the concept definition does not become a part of the individual's concept image.

The lack of coordination between the concept image developed by an individual and the implication of the concept definition can lead to obstacles in learning. This has been corroborated by the work of several researchers. See, for example, Cornu (1991) and Edwards and Ward (2004).

The influence of concept images in understanding of mathematical concepts has been extensively studied. The general attitude observed in students is that they do not consult the definition to resolve conflicts as they do not understand either the relevance or the importance of definitions. (Alcock and Simpson, 2002).

There is also wide literature on many aspects related to issues concerning psychology of mathematical thinking. (Hadamard, 1945, Lakoff and Núñez, 2000). However there is no specific work particularly addressing the processes used by mathematicians to comprehend new mathematical definitions. Hence this set the ground for our study.

RESEARCH METHODOLOGY

The main research instrument was our questionnaire. Bearing in mind our literature survey and a small pilot study in which we interviewed two mathematicians, we prepared the questionnaire. The pilot study helped us to be more focussed in preparing our questions. We sent the questionnaire to twenty five mathematicians. Seven mathematicians responded to our study. The research interests of the mathematicians were varied but in pure mathematics. The respondents, which included two Fields Medallists, were mathematicians who had been doing research and teaching mathematics for several years. The communications were through e-mail, telephone and personal interviews.

THE QUESTIONNAIRE

We list below questions in the questionnaire whose responses will be analysed here in detail.

1. How would you comprehend a definition in your area of expertise and a definition in an area less familiar to you? More specifically, is there a specific identifiable intellectual process specific to your individuality which is called up when needed to comprehend a definition. As a part of the process, do you use special examples and then abstract the process, or draw pictures, schematic diagrams, etc.?
2. Do you have a recollection of having understood a definition or a mathematical statement in a particular way which later on resulted in a conflict? If so, how is the awareness of the conflict triggered? Does the awareness occur spontaneously or when working consciously at it?
3. Is it possible to evolve general strategies for understanding mathematical definitions based on your research experience or teaching? To what extent are the strategies common or different across the subjects (algebra, topology, geometry, analysis, ...).

OUR FINDINGS

We have classified our findings into three categories namely the role of examples, strategies for understanding definitions and the role of conflicts. We shall comment on their responses under each category.

1. Role of examples.

All mathematicians without exception who responded to our questionnaire mentioned the important role played by examples in their comprehension of definitions. We give below some sample responses.

Response 1. To comprehend a definition means it usually includes examples, simple counterexamples, and theorems using the definition.

Response 2. Trying to compile a list of "examples": concrete objects satisfying conditions of the Definition. ``Small" objects: groups with one, two, three elements. ``Big objects": integers, rationals, matrices...

Response 3. I try to see how the definition will exclude/include examples that I already know. For example, if the definition is about groups then I would try to see whether it clearly divides the groups that I know into those that fit and those that do not. Knowing enough examples always helps. Having

a theoretical framework in which to fit the examples also helps.

Response 4. In my area of my expertise, it is less of a challenge for me to comprehend a definition, as I may have already built up an intuition in that area, and examples are already swirling in my head upon reading a definition. In an area less familiar, I cannot feel like I understand a definition (or a theorem for that matter) until I have checked myself that it is not an empty definition (or theorem). I immediately try to think of easy examples, try to make a picture or connect it with some definition I already know and see how it differs in comparison. }

Response 5. Examples are scaffoldings as one tries to build one's understanding of definitions. They are the steps to attain higher and higher levels of understanding. They are also the pillars on which the definitions rest. The more examples one has, the more closer one is to understanding the mathematical object.

If you want to go from A to B, there may be several ways. Different examples provide the different routes. In fact examples give approximate shape of the object that is defined. So, complete understanding is impossible. As you make interconnections you get a finer and finer picture of the object.

Response 6. The most significant aspects of a new idea are often not contained in the deepest or most general theorem resulting from the idea, but they are often embodied in the simplest examples, the simplest definitions and their immediate consequences.

Commentary: These responses indicate that examples play a very important role in understanding mathematical definitions. The knowledge of a mathematical object seems to be closely linked with having sufficient number of suitable examples that characterizes the object. The image forming and image having levels of Pirie and Kieren seems to be related to identifying examples and non-examples associated with a definition. By making sense of theorems that follow from the definition, these images are further strengthened. The very fact that examples are like scaffoldings indicate that when understanding grows, one is freed from the need to take particular actions as examples. The abstraction of the definition is unveiled through examples. For these mathematicians intuition in any particular area is also built by having a rich source of examples.

2. Strategies for understanding definitions:

Response 1. Some prefer to start with some examples and then work slowly to an abstract setting. And some much prefer to start with a general abstract definition and find examples as they crop up. I am strongly of the first type and always seek to delineate the abstract concept with an array of examples or else I just can't work with it.

Response 2. I had once a professor of geometry (P. Libois) who told us his main aim was that we understood . . . [I don't remember if it was "cube", "tetrahedron", or "projective plane"] as well as we understand what "chair" means. We can recognize a chair, sit on one, or in case of need stand on one to reach a high shelf.

That is what, for me, it is to comprehend a definition. It usually includes examples, simple counterexamples, and theorems using the definition. It happens that "definition" and "theorem" can be interchanged, and sometimes this makes a better understanding.

Response 3. Understanding (comprehending) a mathematical definition is a process which is in

principle open-ended: you can never tell that you understand something completely. It can be conceived as consisting of several stages.

(i) Stage 1. Understanding the language in which the definition is stated. In mathematics, I will take for granted that this means the language of more or less formalized set theory, which is expressed in metalanguage based on some natural dialect: English, Russian ... All other choices lack universality/conciseness/ common acceptability etc of Set Theory. However, they might be unavoidable at earlier stages of studying mathematics, e.g. if one is taught Euclidean geometry.

(ii) Stage 2. a) Understanding of the Definition itself as a syntactically correct and meaningful expression of the language of Set Theory.

b) Forming imprecise but intuitively helpful "semantic cloud" of the definition. Let's take as a representative example the definition of a group. Its "semantic cloud" consists in various ideas about symmetry: symmetry of "things", symmetry of patterns, symmetry of physical laws ...

(iii) Stage 3. Trying to compile a list of "examples": concrete objects satisfying conditions of the Definition. "Small" objects: groups with one, two, three elements. "Big objects": integers, rationals, matrices ... Can one classify small objects? Describe explicitly groups of 1,2,3,4,5,6 elements up to isomorphism? Here one more definition crops up that of isomorphism to which the same (up to now, three stage process must be applied).

(iv) Stage 4: studying how the Definition works in various theorems about groups, and in various theories where groups are not the central, but an important part of the picture. Where and how we use associativity, existence of identity, existence of inverse element ...? When there is a chance that all these conditions for a composition law will be satisfied, and when not? Does a given theorem remain true if one omits existence of inverse element in the definition? What kind of "group-like objects" we get then? etc.

All stages, but especially Stage 4 is in principle open-ended. It might involve returning to stage 3, posing and solving classification problems, sometimes marvelling at their complexity (classification of simple finite groups). It enriches our grasp of semantics of the basic language and in this way helps to understand further definitions.

Response 4. I try to make a picture or connect it with some definition I already know and see how it differs in comparison.

Response 5. I tend to continually try to mould the definition that I am reading/hearing into one that fits my area of expertise; if not in content then at least in style.

Response 6. When one meets an elegant result in an area, one usually marvels at the proof. if I would like to understand this at a deeper level, I usually try to formulate a question in which these notions would intervene and this way, I learn to those concepts as well as techniques.

If the definition is equivalent, I note it in the back of my mind that this is an equivalent formulation. at times, the alternate way of looking at things proves useful in solving problems

Response 7. In some sense, a mathematical definition is an isolated tool obtained from a mass of concepts which is utilizable again and again. From existing mathematical concepts, when a selection is made by rearrangement so that this rearrangement becomes a useful tool, or, when a part of an

existing concept is isolated so that it becomes an entity in its own right, a new object is born and it is characterized completely by its definition.

Commentary: These responses seem to suggest that encountering alternate definitions and proving theorems based on the definition increases understanding. When a learner encounters a new mathematical formal definition, he develops a concept image associated to it. When this image becomes a useful tool, this concept image may replace the definition or rather it becomes his concept definition. Self-enquiry, or posing questions to oneself as to, what would be the consequences if one removes or modifies a particular condition from the definition etc improves understanding. Connecting the new definition with the one already known by comparing and contrasting also is an tool used for understanding. The various stages involved in understanding given by one of the mathematicians can be compared to the levels of understanding given by Pirie and Kieren.

According to Lakoff and Núñez (2000) it is through *conceptual mapping* by which mathematicians conceptualize mathematical concepts. The domain of the conceptual mapping consists of examples and the target is the algebraic structure which underlies the set of examples. The example of groups allows one to conceptualise the abstract notion of a group. It is through such (unconscious) metaphorical mappings mathematicians assign an algebraic essence to an arithmetic structure.

3. Role of conflicts:

Response 1. I guess it might also sometimes be required to ignore the usual meaning of the word being defined, but this always came naturally to me.

Response 2. When I first learnt algebra, I imagined all the objects constructed there as "discrete" or at best like rational numbers. This led to a conflict with the "ruler placement postulate" which, by "decimalizing" the line seemed to make it discrete. More generally, I had difficulty understanding topological groups, rings and fields as these were (to my mind) "discrete". In some way the categorical/geometric definition of the operations (as opposed to the point-wise definition) has solved this conflict for me. I similarly had a problem with the "point wise" definition of functions which has become clearer now that I see various spaces of functions (say continuous functions) as completions of "standard" spaces such as polynomials.

There was a while when I could not convince myself of the continuity or well-defined-ness of a function except by detailed case by case analysis. Only later did I understand that the definition was not meant to be used "as such".

Response 3. I recall conflicts occurring when I convinced myself that the definition meant a certain something, and then later on I encountered a counterexample! I had to re-evaluate my understanding and re-understand it correctly so that the paradox could be resolved.

Response 4. I remember carrying many conflicts with me which get resolved through years. Some of them being "tending to infinity" and understanding "Real line is infinite".

Commentary: These responses are suggestive of the fact that conflicts also play a predominant role in understanding. Sometimes they remain dormant and colour one's understanding of new knowledge structures. When one receives external stimuli in the form of a new mathematical definition, he tries to incorporate the new knowledge to his existing knowledge structure. This can be influenced by distorted images resulting in a conflict.

DISCUSSION AND CONCLUSION

Based on the inputs from various mathematicians who responded to our Questionnaire, we draw the following general conclusions. These are based on the important aspects the participants of our study shared with us.

- Examples seem to play a predominant role in comprehending new Definitions. This is one aspect stressed by all our respondents. Examples seem to be an important cognitive tool that mathematicians employ during the process of comprehending a new definition. One of the processes employed is based on the 'inclusion-exclusion' principle. To quote one of the mathematicians
“ I try to see how the definition will exclude/include examples that already know. For example, if the definition is about groups then I would try to see whether it clearly divides the groups that I know into those that fit and those that do not.”
- The set of mathematical objects being defined can be viewed as specifying a 'territory'. Objects can be differentiated according to whether they belong to this territory or not. Therefore examples and counterexamples can be viewed as bringing out an approximate form or shape of this territory, the more numerous and varied the examples the finer the approximation.
- Definitions in ones area of expertise are easily understood as test examples are readily available. One has at hand a rich supply of objects on which to test the definition. Thus examples are like pillars on which definitions are built. For some experts, understanding a new definition involves the process of continuously moulding it so that it fits into their area of expertise.
- It happens that "definition" and "theorem" can be interchanged and sometimes this makes a better understanding. A mathematical object is better understood as one learns various characterizing properties of the same. This is indeed an open-ended process and perhaps it is impossible to know all the equivalent properties of a given mathematical object.

Discussion:

Sometimes when a definition is understood as meaning something, encountering an instance of it which violates what was presumed to be valid can lead to transformation in one's thinking and helps in resolving the conflict. For example, one might believe that all continuous functions are differentiable until one sees an example which is continuous but not differentiable. It appears that understanding of a definition undergoes constant changes as conflicts of various kinds are evoked and resolved.

According to one of our respondents, understanding a mathematical definition consists of the following stages. The first stage involves familiarity of oneself with the formalized mathematical language of the definition and the dialect in which it is written. The second stage involves understanding of the definition as a syntactically correct and meaningful expression of the mathematical language and developing an intuitive understanding of the definition. He calls this

“semantic cloud”. It appears that semantic cloud is the mathematical aspect of what is termed as concept image. (Vinner and Tall, 1981). According to Thurston (1994), learning about a mathematical topic consists of building useful, non-formal mental models, and this cannot be accomplished by studying definitions and rigorous proofs alone. We consider these as being part of the semantic cloud.

Pedagogical implications:

We summarize the pedagogical implications derived from the responses of the mathematicians who responded to our questionnaire:

- (1) Teaching strategies must take into consideration the different challenges posed by each of the stages (given above) in understanding the mathematical definition.
- (2) It is preferable to start with some examples and then work slowly towards an abstract setting. It is important to delineate abstract concepts with an array of examples which tie the idea into their cognitive framework.
- (3) The teaching strategy should aim to convey that a mathematical definition is just as tangible as a table or a chair. The student should be able to recognize it, use it for the routine purposes for which it is meant, and perhaps use it in a novel way, just as one can recognize a chair, sit on one, or in case of need stand on one to reach a high shelf.
- (4) Solving well-formulated problems is an important strategy to gain in-depth understanding of the definition.
- (5) Examples are important in every subject and some subjects are so closely related that understanding of definition in one contributes to understanding of parallel and analogous definitions. Such strategies are routinely adopted by professional mathematicians

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