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TACKLING THE DIVISION ALGORITHM

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Based on a three week study conducted with students (n ~ 42) at a government girls school in rural Madhya Pradesh, this paper analyses some of the difficulties surrounding the teaching and learning of the standard long division algorithm, and argues that an alternate approach based on sharing makes for a more flexible and effective teaching of division. The work done is preliminary, but offers insights into developing an alternate teaching-learning trajectory for division beginning at the primary school level.

Keywords: sharing, partial quotients, standard long division algorithm, decimal system of representation

INTRODUCTION

The teaching and learning of division has been a topic of much debate in the last few decades. Most government schools in India following State textbooks introduce the standard long division algorithm soon after a cursory introduction to the concept of division. It has been found however, that several students struggle to use the algorithm correctly, and of those who do, many perform the correct steps without relating the taught procedure to the concept of division or their everyday experiences of sharing and grouping. The paper hopes to highlight some of the difficulties surrounding the teaching and learning of the standard division algorithm and describes the outcomes of our teaching experiment that attempted to tackle them.

BACKGROUND AND MOTIVATION

The students we worked with belonged to Grade 6 (aged between 10 and 15 years) of a Government run Girls School in rural Madhya Pradesh, India. The students were part of a three year longitudinal study conducted by Eklavya[1] in the teaching and learning of fractions. There came a point in this teaching-learning trajectory – in using equivalence to compare fractions by reducing them to their simplest form – at which they required the knowledge of factors and hence division. At another point, while carrying out number board activities which called for the use of division, we found that students attempted division using the standard algorithm they had been taught in their regular classes. However, most of them struggled to use it correctly, and we encountered a whole host of errors we wished to understand. It was at this point that we decided to 'tackle the division algorithm'. Further impetus was provided by a string

1 Eklavya is not-for-profit, non-governmental organisation working in the field of education in Madhya Pradesh, India. (www.eklavya.in)

of lively discussions on *activemath*, an online forum dealing with issues of mathematics education in India.

What is it that we are tackling? – The justification for an alternate approach

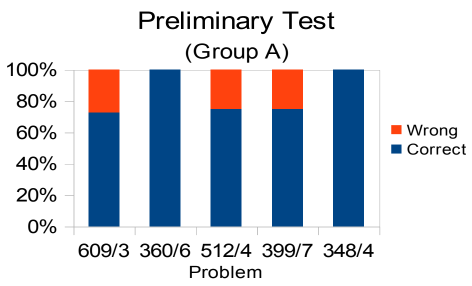
We believe, (as much research on teaching division suggests) that the standard long division algorithm, based fundamentally on the place value system, can be counter-intuitive to children's existing whole number sense. What's more, introducing division solely through algorithmic procedures reduces the process of division to a series of mechanical, unthinking steps which makes it difficult for a student to relate the taught procedure to the meanings that can be identified with division, such as sharing and distribution. [Anghileri, J. et al (2002)]

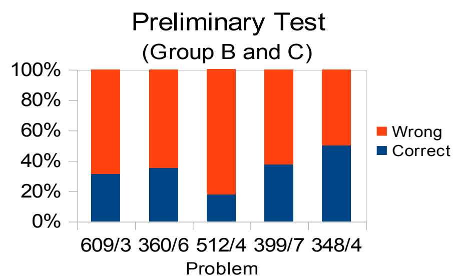
Historical and contemporary considerations: A look at the literature

Much of the teaching of arithmetic in the early part of the century was characterised by the belief in drill and practice that favoured computational skill over conceptual understanding. [English, L. (1995)] However, more recently, several studies point to the fact that teaching division concepts through the algorithm makes it difficult to grasp the mathematical concept of division, and results in a host of confusions. Windsor, W. and Booker, G. (2005) point to the fact that a historical analysis of the division algorithm might provide insights for improved teaching methods, and attribute the 'natural origin' of the 'ineffective teaching strategies' to the understanding provided for division procedures by the Egyptian, Chinese and Hindu-Arabic number systems. They assert that the use of 'inappropriate language' (such as the 'bringing down' of a digit, and 'putting a number into another') confuses students and hinders the understanding of the division concept. They feel that the language of division should be established through support of materials and children should have ample 'division experiences' before working with the algorithm. Leung, I., Pang, W., and Wong, R. (2006) contest that the 'guess-and-match type mental process' of maximising the quotient confuses the concept of division and is not consistent with children's life experiences associated with grouping and sharing. Initial results of their studies suggest that children taught by a new method, very similar to ours, perform better on a test than those who learn it through the traditional method. Anghileri, J., Beishuizen, M., Van Putten, K. (2002) conducted a comparison study between Dutch and English children to explore their written calculation methods for division, and found that Dutch students, who had been taught division through a 'careful progression' of informal strategies to a more structured, efficient procedure, performed significantly better than the English children who have been mostly taught the steps of the standard algorithm.

Cues from the classroom

We hoped to introduce an alternate approach in such a way as to draw a parallel with the standard long division algorithm. In order to gauge their understanding of and comfort with the algorithm, we conducted something of a preliminary 'test' that shed some light on the meaning they associated with the algorithm. The exercise consisted of five division problems with a three-digit dividend and a single-digit divisor, designed to cover the most common types of errors children are known to commit while using the algorithm. The problems given were 609 ÷ 3, 360 ÷ 6, 512 ÷ 4, 399 ÷ 7, and 348 ÷ 4. Around 12 students seemed reasonably comfortable with the algorithm. Their results are shown in graph 1. The remaining students seemed to struggle with the algorithm, and as graph 2 shows, in all five problems at least 50% of the solutions



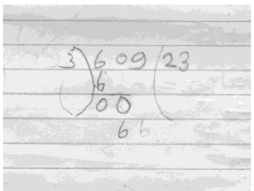


were incorrect.[2]

Fig. 1

Fig. 2

Below are some examples of students' errors:



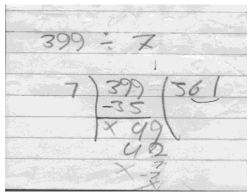


Fig. 3

Fig. 4

Errors such as these helped us identify some of the difficulties that surround the standard algorithm, and served as cues to work with an alternative approach which has been use earlier by the Delhi SCERT textbooks and is currently being used by the NCERT as well.[3]

DESCRIBING AN ALTERNATE APPROACH

We had roughly divided the class into four broad categories (A, B ,C and D) based on what we believed their level of mathematical understanding and skills to be, with students belonging to category A being the most competent, and those belonging to D being the least. This has been done informally, based on our interaction with them, in order to aid classroom transaction, as it was becoming increasingly difficult to address the varied levels of competency within the same class. Since, students of category D struggled to even recognise number symbols or pick number quantities greater than 30, we have refrained from including these 10 students from the bulk of our analysis.

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2 We introduce a grouping later in the paper, for which this differentiation proves insightful. The 12 students constitute Group A while 'the rest' constitute Group B and C. Refer to the methodology and results section.

3 NCERT Class 5 textbook, http://ncert.nic.in/NCERTS/textbook/textbook.htm?eemh1=0-14

Over the course of the study, classes were held for a period of a little more than an hour. A regular diary of classroom observations was maintained. Students were given worksheets regularly and did not have any time limit. Other written work and exercises were done in a notebook given to each student at the beginning of the class, which was then taken back once the class was completed.

After the preliminary 'test', we began by relating division to the activity of sharing. Using matchsticks bundled into groups of 10[4], students were asked to pick a given number of matchsticks less than 100 at random (serving as the dividend) and then pull out a given number of empty matchbox trays (serving as the divisor). While distributing the matchsticks equally among the trays, each student would simultaneously record her steps in a representation resembling that of the algorithm (See Fig. 5). The number of matchsticks found in each of the trays at the end of the process represents the quotient, while those that remain undistributed – if any – represent the remainder.

4	98	10
	- 40	+
	58	10
	-40	+
	18	4
	-16	
	2	24

Fig. 5

Later, we urged students to use this representation to solve division problems without the support of material. In drawing on the process of sharing and the idea that multiplication is distributive over addition, this method allows for a decomposition of the quotient into parts that are chosen by the students themselves, which will eventually add up to be the quotient (as in the example above). This method is commonly referred to as the partial quotients method.

Since the students had already been introduced to the standard algorithm, we attempted to work with both methods in parallel, trying to establish a link between the two. We introduced a currency note model to explain the standard algorithm.

RESULTS

We discuss five different aspects of the results of our study.

Meaning attributed to division

Below is an example of one of the students' work, which perhaps reveals the difference in the meaning they associate with each of the methods.

4 We hoped this would be a way of appealing to the students existing knowledge of the place value system, without addressing it in a formal manner. Note that the bundling was done by the students themselves.

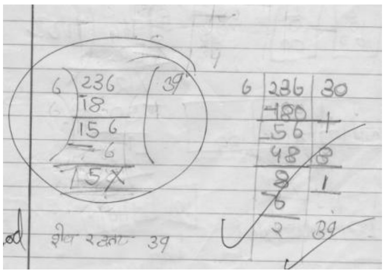


Fig 6

Using the standard long division algorithm, the student doesn't seem to demonstrate a grasp of the procedure or its link to the meaning of division. She considers 23 of the 236 first as demanded by the algorithm, and finds the maximum multiple of 6 that is less than or equal to it. A subtraction error (that also perhaps arises out of a lack of conceptual understanding of subtraction in using the column wise subtraction algorithm[5]) leads her to write 15 as the difference between 23 and 18. She brings down the 6 as she has been instructed to do, places 9 beside 3 in the quotient and subtracts 6 from 156, getting 15X. The 'X' is something students are taught in association with the division algorithm, as a symbol for 0 while performing a subtraction. It is unclear as to why this is so, but it is practiced widely by teachers. It is uncertain what the student understands by 15X-whether 150 or 15. The informal impression we have, however, is that if she were to perform the subtraction problem of 156 – 6 in isolation, she would write 150. As a stark contrast to this seemingly insensible solution, she has clearly and accurately managed to divide (and subtract) using the partial quotients method, giving her that 236 divided by 6 yields 39 with remainder 3.

Another aspect of meaning production lay in the possibility offered by the PQ method for students to distribute in multiple ways. Algorithms are based fundamentally on a set procedure, and a failure to comply with the steps will automatically lead to a 'wrong' answer. Contrarily, the multiple possibilities offered by the partial quotients method, enables students to associate a host of meanings to the concept of division.

Ability to distribute vs the ability to divide

It is perhaps necessary to distinguish between the activity of distributing n objects equally among m places and to be able to divide the number n by the number m and maintaining a formal written record of it without the use and material. When we began the activity of distribution, we found that several of the students who were unable to use the algorithm correctly, were in fact able to distribute matchsticks among trays with much ease and deftness. For numbers less than 50, even students of Group D were able to distribute matchsticks into trays rather efficiently, by which it is meant, they tended to choose reasonably large chunks to place in each tray, often just one less than the maximum possible.

5 Kamii, C., & Dominick, A. (1997) contest that the column-wise algorithms serve to 'unteach' place value and discourage students from developing number sense

For all students however, when supplied with material, writing out their steps in a formal representation introduced by us seemed rather unnecessary since they had just demonstrated the whole act of division using the objects in front of them. We believe that the transition from the concrete to the abstract, however, requires time, and that the intermediary step of working without material, and instead evoking meaningful contexts to solve problems and formally record their solutions, is crucial to making this transition. Since we were rushing a whole set of experiences into a short span of time, it became difficult for some students to mature from one phase to the other.

Fig. 7 draws distinctions between the act of distribution, the act of solving a division problem on paper using the partial quotients method, and that of solving a problem using the long division algorithm:

The physical act of distribution	**Solving a problem using the PQ method**	**Solving a problem using the long division algorithm**
• No formal representation • Does not demand the knowledge of subtraction, eliminating the scope for subtraction errors • Grants students the freedom to choose quantities for distribution • Usually time consuming and inefficient	• Involves formal representation • Relates in a natural way to students intuitive notions of division • Demands the knowledge of subtraction • Grants students the freedom to choose chunks of their choice • Could be less efficient that the algorithm	• Involves formal representation • Does not relate in a natural way to students intuitive notions of division • Is counter-intuitive to students knowledge of place value • Demands them to execute subtraction in a way very different from what they have been taught so far. • Does not allow choice of multiple. Demands that they find at each step a maximum multiple of the divisor so that it is less than or equal to the dividend/virtual dividend • The most efficient method

Fig. 7

A reduction in errors

We classify the most common errors into the following broad categories:

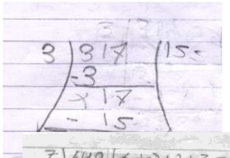
Place value errors: The division algorithm demands that students do not treat the dividend as a whole number, but instead focus on its digits. For example in dividing 360 by 6, (see Fig. 9) a student is required to look at 36 and find the maximum multiple of 6 that is less than or equal to it. This multiple then, is a *virtual quotient* that forms the leading digit of the actual quotient. The 36 serves as a virtual dividend.[6] This focus on digits counters students' existing knowledge of the place value system, as a result of which, as in the examples below, 0 has lost its role as a place holder. What's more, the subtraction inside the working of the algorithm is unlike what they have learnt before – it becomes partial, since you are required to take away the 2-digit number 36 from the leading two digits, 36, of 360 (see Fig. 9) and results in a whole host of errors and confusions. For the first time, students are now required to work through the algorithm from left to right instead of right to left as they had been taught to do in the column-wise algorithms of addition and subtraction. This treatment of

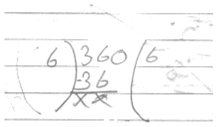
6 The terminology of 'virtual quotient' and 'virtual dividend' has been used by Leung, I. et al (2006)

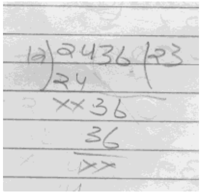
numbers is counter-intuitive to their whole number sense, causing something of a cognitive conflict – one that does not aid learning, but rather, serves to confuse without providing any insights.

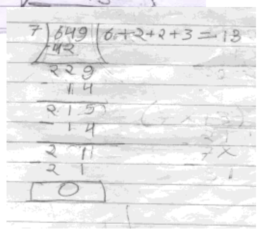
Fig. 8 Fig. 9

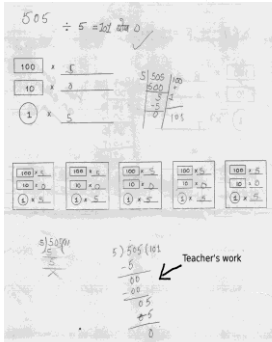
In Fig. 11, the student switches from subtracting the virtual product from the virtual dividend, to simply subtracting a multiple of the divisor as a whole number, which

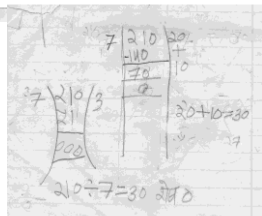












corresponds with what she has been taught earlier about subtraction, but leads to a violation of the algorithm. At the end, once again we see that she exhibits confusion about place value, and 1 loses its place value, giving that 215 – 14 is simply 21.

Fig. 10 Fig. 11 Fig. 12 Fig. 13

These kinds of errors are tackled in the partial quotients method since students are working with whole numbers at each stage of the division process. In Fig. 12 the same student gets 505 divided by 5 as 11 using the standard algorithm and 101 using the partial quotients. Similarly in Fig. 13 the student writes 210 divided by 7 as 3 using the algorithm and as 30 using the partial quotients method.

Errors arising out of the problem of maximising the quotient: One of the main demands of the division algorithm is the maximisation of the virtual quotient. This task of maximisation doesn't allow students to work with a limited knowledge of multiplication facts. Further, the expression used by teachers to convey this rule, is often interpreted by students such that there is no stress on the *maximum*. This compounds these kinds of errors. Examples from students' work are seen below.

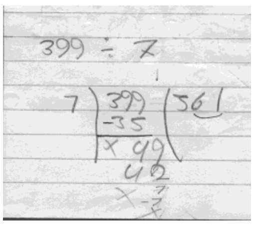


Fig. 14

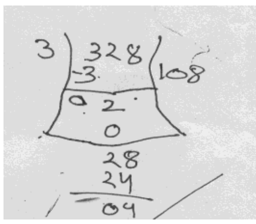


Fig. 15

In the partial quotients method however, students can draw on multiplication facts of their choice, thus reducing the scope for errors.

The success of the partial quotients method is best reflected in Groups B and C. Most students from these two groups (as we saw in earlier data[7]) were struggling to use the algorithm correctly before the introduction of this method. While we recognise the limitations of quantifying data, we believe the following charts provide some insightful information regarding the errors related to each method and the comfort and progress of students with the partial quotients method.

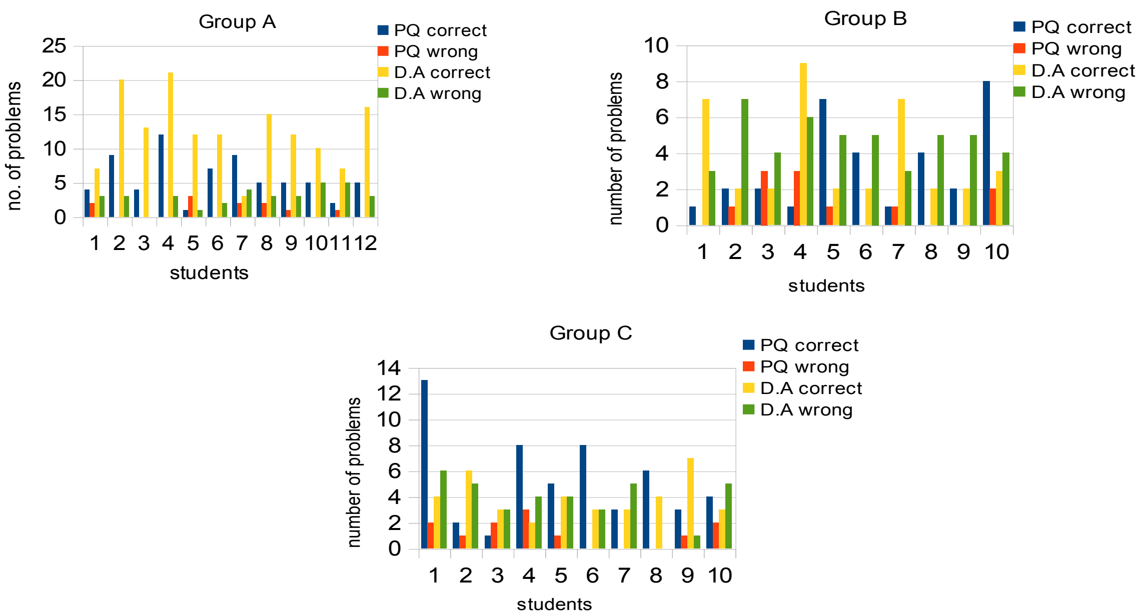


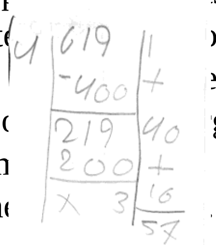
Fig. 16

The charts record four data types for each student – the number of problems solved correctly using the standard division algorithm (DA correct) and the number solved incorrectly (DA wrong), and the corresponding numbers using the partial quotients method (PQ correct, PQ wrong). The data has been collected over the period of three weeks from all their written work; it does not refer to one specific set of problems. What we see is that students of Group A have solved several correctly in both methods (the numbers are higher for the division algorithm as they were encouraged to use it more than other students) and that the errors with partial quotients are far less than those with the algorithm. For Group B, 4 out of 10 students have committed no mistake at all using the partial quotients method. Errors with the standard algorithm are significant. In Group C, all except for two students have more incorrect solutions using the standard algorithm than correct solutions, and in using the partial quotients method, incorrect solutions are significantly low.

7 See 'cues from the classroom' and footnote 2

The link between the two methods and the partial failure of the note model

We hoped that, once the students were comfortable with the partial quotients method they would be guided into maximizing chunks so as to see the link between this method and the long division algorithm. Some students used the results of the standard algorithm to solve the problem using the partial quotients method – choosing the final quotient as the first chunk to be distribut[...] others carried out the partial quotients method with the logic of the [...] (see Fig. 17 and 18). However, while most students recognised early [...] larger the chunks they distribute, the faster they would solve the problem [...] students were really able to establish a clear conceptual link between th[...]



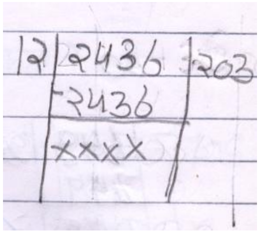


Fig. 17

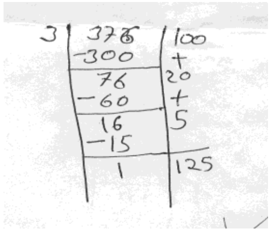


Fig. 18

Unless forced into working with a decimal based maximizing of chunks, the partial quotients method doesn't naturally lend itself to arriving at the algorithm. We therefore tried, in parallel, to introduce a currency note model to explain the standard algorithm, emphasising that if we were distributing money among children, then the partial quotients method dealt with the quantity of money to be distributed, while in the algorithm, we looked at the number of notes (of denominations 1, 10 and 100) to be distributed. However, these are really just two different models, and nothing in how they were presented demonstrated the conceptual link between the methods.

The challenges posed by this explanation of the algorithm were multiple. In stressing the number of notes to be distributed, the virtual products that form the digits of the quotient actually serve as counters recording the number of notes given away. The quotient then becomes a string of individual counters and it requires an additional step to decode them and read the string as a whole number representing a certain quantity. So while the note model associates a meaningful real life activity to the algorithm, it only helps to explain the steps of the procedure, which continues to be a mechanical process that requires decoding.

Problems of hybridization

Working with both methods simultaneously in such a short span of time resulted in a host of outrageous hybrid solutions such as these:

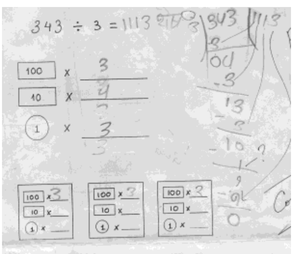


Fig. 19 Fig. 20

We believe, however, that introducing the partial quotients method to students who have not been introduced to the algorithm, would do away with this problem.

CONCLUSION

Building upon students' intuitive notions of sharing makes for a more effective learning of the division concept. Working with the standard algorithm alone denies students a chance to relate the mechanical procedure to meaningful contexts. Our initial study suggests that the partial quotients method enables students to carry out division with meaning and much fewer errors. There is a strong need to develop a teaching-learning trajectory based on such an approach and conduct a systematic study to assess its effectiveness.

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