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STUDENTS SOLVING INVESTIGATORY PROBLEMS IN A DYNAMIC GEOMETRY ENVIRONMENT - A STUDY OF STUDENTS' WORK IN INDIA AND SWEDEN

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The impact of one laptop per child movements around our world will of course affect every school subject, including mathematics. With a computer and suitable mathematical software, students can explore mathematical concepts and make conjectures in a quite new way. During a joint India-Swedish mathematical education conference in Mumbai 2011, the authors of this article met and decided to make a comparative study on how students in grade 7 and 9 solve exploratory problems when using the dynamical geometry environment GeoGebra.

Investigations, technology, quadrilaterals, Euler line.

INTRODUCTION

Increasingly, in many countries across the world, the use of technology in mathematics instruction at different levels has been the subject of many research studies. The number of research journals with the label technology in its title is increasing. One of the key issues confronting mathematics educators is the cognitive transition from experimental mathematics, that is, verification and conjecturing, to theoretical mathematics, that is, formal abstract concepts and proof - an elusive process that many mathematics education researchers have tried to explore. Dynamic technology, allowing the movement of an object that is represented technologically, often affecting the movement of another connected object, transforms the possibilities for representation and may have a great impact on how mathematical objects are conceptualized and mathematical meanings are internalized (Falcade, Laborde, & Mariotti, 2007; Moreno-Armella, Hegedus, & Kaput, 2008)..

THE STUDY

Comparative studies basically search to identify and explain differences of homologous phenomena in two or more contexts. This study aims to investigate and compare the ways in which middle school students in India and Sweden, construct and develop geometrical understanding in a dynamic geometry environment (DGE). The emphasis is on exploring how students make conjectures in a dynamic geometry environment and on the kind of mathematical reasoning they use to explain their DGE explorations. In fact the researchers tried to deliberate on whether a DGE can change the nature of student's geometrical explorations, when compared with paper pencil explorations, and also reflect on the way students perceive proof.

In India, mathematics education at the school level emphasizes use of procedures and methods and a large proportion of the student's time is spent on practising and perfecting manipulative skills. Although technology has pervaded many aspects of the Indian society, in the large majority of schools across India, technology is not used in mathematics instruction. Very little time is spent on visualization or exploration of concepts. In Sweden however, mathematics education in compulsory school emphasises the connection between mathematics and its use in daily work. Hardly any schools use computers for mathematics instruction, although many schools are driving one laptop per child projects. Given the different approaches to mathematics learning in both countries, the researchers felt that it might be worthwhile to see how students of the same grades respond to using a technology tool in exploring geometrical concepts. GeoGebra, a dynamic geometry environment, was chosen as the technology platform for the study. The objective of the study was to reflect upon the following research questions

1. How does the dynamic geometry environment (DGE) help to facilitate the process of making conjectures?
2. Does the process of arriving at a formal proof of a result become a natural extension of conjecturing followed by verification process?

THEORETICAL FRAMEWORKS

Traditionally, geometry is taught and learned in a pencil and paper environment. However, a conceptual understanding of geometry requires mental imagination, since the proofs and derivations of formulas are based on flexibility and generalization of figures or shapes. Textbooks are unable to highlight the dynamic nature of geometrical figures thus leading students to mentally investigate properties of geometrical objects and related concepts without providing an opportunity for manipulation and exploration. However a Dynamic Geometry Environment provides a dynamic and visual representation of geometry concepts in a physical sense. Learning geometry with a DGE might offer students possibilities to construct and manipulate geometrical figures and do empirical investigations. These activities are hard in a static geometry environment (Laborde, 1999).

In particular the researchers would like to investigate how students react when these geometrical concepts and objects are represented in a dynamical geometry system. Following this thread we decided to use the framework proposed by Vinner and Tall (1981), which says that with each mathematical concept is associated a concept definition and a concept image. This framework is natural and quite easy to understand.

The concept definition can be the stipulated as a definition assigned to a given concept. Let us say that we like to define a circle with centre in (a, b) and radius r as the algebraic definition $(x - a)^2 + (y - b)^2 = r^2$. The concept image of that circle, on the other hand, will be a nonverbal representation of any individual's understanding of the concept circle. It includes the "visual representations, the mental pictures, the impressions and the experiences associated with the concept name" (Vinner, 1999, p. 68). Most, if not all, humans probably have a rich experience of the concept circle which fits into the concept image.

According to Vinner, to acquire a concept one has to form a concept image for it. Merely learning the definition of a concept does not guarantee that the concept itself will be acquired. Many mathematics instructors expect their students to develop a concept image based on a definition of the concept delivered in class or from a definition given in a textbook. This, however does not always happen. Vinner also refers to the concept image as an 'evoked concept image', that is, the mental image or a memory that is evoked in the individual's mind when the word related to the concept is heard by the individual. The definition of the concept may have no or little role to play initially but later may become functional once the concept image is formed. For example, the word 'triangle' may evoke the concept image of a geometrical figure of a triangle in an individual rather the definition of a triangle 'a figure with three sides and three vertices'.

An individual may require several inputs (other than a definition) to help form a concept image. For example to help an individual understand the idea of a triangle, the teacher may present several drawings or pictures of triangles, expecting the learner to 'see' some common features (invariant properties), among all the figures, namely, three sides, three vertices etc. The same idea may be explored in a dynamic geometry environment by drawing a triangle and dragging one of the vertices to create an animation, a continuous process, through which the learner will be able to see the invariant properties. Extending this a little further, we may want the learner to conjecture that the sum of the three angles of a triangle is 180^o. Presenting several triangles on paper and making them measure the three angles could be one way. In a dynamic geometry environment, however, by dragging the vertices of a triangle the learner can 'see' in the algebra view, that the sum remains 180^o irrespective of the shape of the triangle. In this study we hope to highlight the fact that exploring a concept in a dynamic geometry environment can play a vital role in shaping the student's concept image.

Sfard and Linchevski's (1994) theory of reification and Sfard's (1991) theory of process/object duality scaffolds Talls and Vinners theory of Concept definitions. . The theory of reification posits the existence of three stages of concept formation: (a) interiorization; (b) condensation; and (c) reification. A significant question involving this theory is – What spurs the transition from one stage to the next? The first two stages represent the operational aspect of mathematical notation and the last stage its structural aspect. According to Sfard (1991) the structural conception of a mathematical notation is static whereas the operational conception is dynamic and detailed. To understand the structural aspect of a mathematical concept is difficult for most people because a person must pass through the ontological gap between the operational and structural stage. Sfard (1991, 3) distinguishes between the words "concept" and "conception". According to Sfard the term concept represents the mathematical, formal side of the concept and conceptions the private side of the concept.

This seems close to Tall's and Vinner's (1981) theory of concept image and concept definition. They suggest that when we think of a concept something is evoked in our mind. Often these images do not necessarily relate to a concept definition even if the concept is well defined theoretically. The collection of conceptions is called the concept image. In Tall's and Vinner's theory, concept image is the whole cognitive and mental structure which is associated with the concept. It is built from the first time we encounter a new concept and is changed when we meet new stimuli regarding the concept in new situations.

Tall & Vinner accept that a concept image might be well rooted in our mind, but nevertheless totally wrong. So they differ between a formal concept definition as a definition accepted by mathematicians, physicists or other experts and contrast this with the so called personal concept definition which might be the phrase someone uses when he/she is asked to define a concept at a certain moment and which might or might not be coherent with the formal definition.

> For each individual a concept definition generates its own concept image (which might, in a flight of fancy be called the ”concept definition image”). This is, of course, part of the concept image. (Tall & Vinner 1981)

A student who has experienced that a parallelogram always has one side parallel with the top side or bottom side of the text book might have a limited personal concept definition of what a parallelogram is. We will see how these two frameworks play out in our description of students work.

METHOD

The subjects of the study were middle school students from India and Sweden. In India 28 students from grade 7 and 22 students from grade 9 participated in the study. In Sweden there were 24 students from grade 7 and 28 students from grade 9. The dynamic geometry software chosen for this study was GeoGebra. In India the students were given access to GeoGebra in a computer laboratory. In Sweden the students brought their own laptops with them to the sessions and some students had to download and install GeoGebra during the first few minutes of the first class session. There were two 2 hour sessions with the grade 7 groups and similarly two 2 hour sessions with the grade 9 groups in India at one occasion, while the Swedish students were selected from three different schools in Gothenburg in order to get students from different socio-economic backgrounds. In each school grade 7 and grade 9 students underwent 2 two hour sessions each. At the end of the sessions an attempt was made to lead them to construct the proofs of some important results.

The subjects of both countries had no prior experience in using GeoGebra and were assigned tasks in the form of investigatory problems and constructions in geometry related to topics of the curriculum. The tasks were designed keeping in mind a constructivist perspective and students were required to try out these tasks on GeoGebra with the authors facilitating the process. These were presented to the subjects in the form of worksheets. The subjects were encouraged to perform constructions using GeoGebra, make conjectures based on their observation and finally verify these conjectures. They were required to write their responses in the worksheets. We saw this setup as an important part of the study. Most compulsory teaching in mathematics is probably rather standardized and little time is devoted to guessing, seeking patterns or making and testing conjectures which leaves very little possibility to do new mathematics.

Sweden

All Swedish students were from ordinary community compulsory schools. The Swedish grade 7 students had an average age of 13 years and the grade 9 students had an average age of 15 years. With respect to the socio-economic background of students, of the three schools, all in Gothenburg, one was in a wealthy area, another in a poor area and the third one was in a middle class area. No significant difference in performance could be observed between students from different areas of Gothenburg. All three Swedish schools were in a one laptop per child movement and all Swedish students were computer experienced. Grade 7 students were familiar with the concept of quadrilaterals and their basic properties and the grade 9 students were familiar with the concept of a triangle's centroid and of measuring in triangles. They had no experience in exploring the orthocenter, the circumcenter or the incenter.

In all three schools one researcher (R) was present all the time in a regular classroom and along with the teacher in both grades 7 and 9. The students brought with them their own laptops, since all schools were part of one laptop per child project. All three sessions started with R introducing the project to students as a collaborative endeavour between India and Sweden. R also explained that the exercises were in English and hence students could use Google translate in order to understand words as quadrilateral, vertices, perpendicular, and so forth.

The students were grouped in groups of 3 or 4 students and R asked for groups who volunteered to use Screen cast technology to record their work. The address for ScreencastOmatic was given to the students. One of the schools had problem with their WiFi net, so the screen cast exercise was dropped at that location. The GeoGebra activities on the student's monitors were filmed without the students due to ethical restrictions. Two video cameras were used at every occasion, one for the 7th graders group and one for the 9th graders group. In each 7th and 9th graders group, 3 or 4 students accepted to use the screen casting facility in Screencast-O-Matic, see http://www.screencast-o-matic.com/. These students used screen casting for one of their constructions every 40 minutes or so, where they created a mp4 files of 5 minutes duration. All students used the possibility to support their screen cast with oral support. Through this method one can see each step taken by the students take in a construction. During the whole session, a minimum of information was given unless some of the students asked direct questions about the meaning of a sentence or if he/she was right about a conjecture. At the end of the sessions all students turned in their written report and the voluntary groups saved their screen cast movies on R's USB stick (in two schools).

India

The 28 Indian students of grade 7 and 22 students of grade 9 were selected from a set of three schools in Mumbai, India. The students of these schools were from middle class families and most of the parents of these students were scientists by profession. These schools follow the curriculum prescribed by the State board, which does not prescribe the use of technology for teaching mathematics or permit its use in examinations. Although schools have the freedom to integrate technology in their classrooms, this was not the practice in the selected schools. Students of grade 7 were familiar with the concept of quadrilaterals and their basic properties though they were not familiar with the midpoint quadrilateral problem. Also most of the geometry was taught to them were in the traditional paper and pencil manner. The same was the case with students of grade 9. They were familiar with the definitions of circumcenter, orthocentre and centroid but had not explored them. They knew how to construct the perpendicular bisectors, altitudes and medians of a triangle using ruler and compass but hadn't explored the same using paper folding or any geometry software.

The two researchers (R1 and R2) conducted the sessions in a computer laboratory. R1 began the session by talking to the students about the project, emphasizing the fact that their responses would be valuable in the research project. R2 took rounds helping the students. Students were sitting in pairs at the computers. Both R1 and R2 gave an introductory overview of different tools in GeoGebra in order to familiarize students with GeoGebra.

RESULTS

Sweden

A general observation was the ease with which students approached the tasks in GeoGebra. This underlines the fact that students of today have a general digital competence. Interestingly, however there were some students in both the 7th graders and in the 9th graders group, who considered the work in GeoGebra to be a sort of "cheating" and the work with paper and pencil more the "real mathematics". This is probably a result of the fact that their teachers had not used GeoGebra to show constructions in class.

All 7th graders finally reached the conclusion that the inner construction of the quadrilaterals was a parallelogram and all of them also confirmed the correlation between the slopes. Nevertheless, there is a difference in the conceptual image of the parallelogram and of the slope. The slope is mainly there as a tool to take you further, whiles the parallelogram is a thrilling result. Only 13 of 24 managed to measure the area and reach a conclusion about the ratio. Of these 13 7th graders, 5 students managed to form a general statement regarding the ratios of the areas of the inner parallelograms. All these 5 students also managed to perform a logical reasoning connected to that statement, a sort of homemade proof based on triangles. When doing so, they all used paper and pencils and the figure in the material they were given, sometimes supported by GeoGebra. One might say that they partly abandoned GeoGebra when it was time to do a proof.

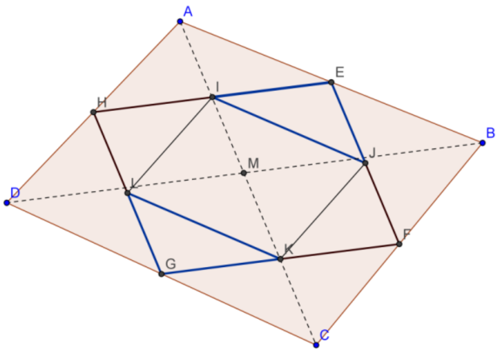

Figure 1: Even 7th graders can do informal proofs by indentifying congruity.

Most of the students drew a sketch on a paper, similar to the sketch in figure 1. The reasoning was often supported by a movement in GeoGebra. This is a dialog between two students in grade 7:

> Student 1: In my diagram (the student draws a segment from H, through M and to F) I can see that I have two triangles here, one I label EFH and one I label EIJ. The bases are parallel, so EFH is just an enlargement of EIJ. Since I is on the midpoint of EH, the area of EIJ must be ¼ of the area of EFH.
>
> Student 2: Yes, I see what you mean and I can agree with that. But how do you explain that the inner quadrilateral is a parallelogram?
>
> Student 1: Well, let's see. Since EI is the half of EH and since EJ is the half of EF, then IJ must be parallel to FH. That holds also if we turn the sketch 90°and if two opposite sides of the figure are parallel, then the figure must be a parallelogram
>
> Student 2: Show me once again. Yes, no I see. Isn't that amazing, that it always is a parallelogram, regardless of the outer quadrilateral? I have never thought of a parallelogram like this before.

Theoretical analysis. If we look at the exercise with the midpoints of a quadrilateral, the first step is to construct a dynamic quadrilateral in GeoGebra which can be moved around. We see this as interiorization, where students deal with a familiar object - the quadrilateral. The next step is to construct an inner quadrilateral from the midpoints of the outer quadrilateral. We see this as condensation, since the inner quadrilateral is a separate entity which can be manipulated by dragging the vertices of the original quadrilateral. Finally, when students understand that the inner quadrilateral always is a parallelogram, despite the appearance of the outer quadrilateral, then they see the whole construction in a new light and we see that as reification of the object parallelogram. This is an 'ontological shift', a sudden ability to see a familiar object, which is the parallelogram as a product of the process of joining the midpoints of any given quadrilateral. In another group the following discussion took place. Once again, we find one students trying to tutoring or at least convincing the other student.

It stands out as quite surprising that Swedish students who never have seen or perhaps even heard of mathematical proofs strive for correctness and rigor in that sense. Is that part of the work with the instrument itself? Is this a driving force inside the work with constructions?

> With the emergence of dynamic geometry software, Euclidean geometry in general, and later theorems in particular, have aroused renewed interest. In these 'microworlds', geometrical theorems can become much more than propositions waiting to be proven, they can become projects for investigation, which rely on the ease with which many instances of a proposition can be obtained, analyzed, measured, and compared. All these at the service of conjecturing and testing (empirical 'proof'). (Bruckheimer & Arcavi, 2001)

The Euclidian curriculum disappeared in most curriculums many years ago after many years of debate. Yet, the long going discussion still echoes over the last decades.

> The concept of proof is fundamental in mathematics, and so in geometry the students have the opportunity to learn one of the great features of the subject. But since the final deductive proof of a theorem is usually the end result of a lot of guessing and

> experimenting and often depends on an ingenious scheme which permits proving the theorem in a proper logical sequence, the proof is not necessarily a natural one, that is, one which would suggest itself readily to the adolescent mind. (Kline, 1973, p. 7)

However the response of the Swedish student seems contrary to the above. The midpoint quadrilateral exploration led to a situation where students sought a more rigorous proof after they had stated conjectures in the constructional phase of their work. While dynamic geometry software cannot actually produce proofs, the experimental evidence it provides users with produces strong conviction, which can motivate the desire for proof. (King & Schattschneider, 1997, p. xiii). The result from the 9th graders is excluded, due to page limits.

India

In the midpoint quadrilateral exploration, students drew the midpoint quadrilateral EFGH of a given quadrilateral ABCD. R1 and R2 interacted with students and asked them if they recognized the midpoint quadrilateral. There were various responses to this. Some students said it's a square, some said rhombus, rectangle and some said it's a parallelogram.

> R1: Is it a rectangle or a parallelogram? Is a rectangle a parallelogram? Is a square a parallelogram?

This led to an interesting discussion in the class where the properties of squares, rectangles and parallelograms were recalled. After this, students made the observation that the midpoint quadrilateral EFGH is a parallelogram but the type of parallelogram depends on the quadrilateral ABCD. R1 and R2 then asked students to verify that EFGH is indeed a parallelogram. Using the algebra view they observed that opposite sides are equal. Now the problem was to prove that opposite sides are parallel. A few suggestions came from the class but one student suggested 'through H draw a line parallel to FG. If it overlaps HE then HE is parallel to FG'. This was a striking suggestion and many students concurred. The same approach was used to prove that the sides HG and FE were parallel. The students did not know the concept of slope so it was not used by them to verify that the opposite sides of EFGH were parallel.

This is similar to the case of the Swedish 7th graders. The students began to 'see' a parallelogram as a product of the process of joining the midpoints of any given quadrilateral and also discuss its properties. This supports the theory of reification where there is an 'ontological shift', a sudden ability to see a familiar object, (in this case the parallelogram) from a different perspective (in relation to the original quadrilateral). Also dragging the vertices of ABCD, helped students to see the invariance of the properties of EFGH and conjecture that it is a parallelogram. Thus GeoGebra was able to give a physical manifestation to the 'mental animation' which led to the observation of the invariant properties of EFGH.

In the next part of the exploration, the proof of the midpoint theorem was discussed by R2. This was done in the traditional manner through demonstration and discussion and GeoGebra was not used for the proof.

> RI: If we join the diagonal AC of the quadrilateral ABCD can we use the midpoint theorem to to conclude anything?

After some facilitation students were able to apply the midpoint theorem to the triangles ABC and ADC (to conclude that EF and HG are both parallel to AC and equal to half its length) to show that EFGH is a parallelogram. Thus students dragged the vertices of their figure in GeoGebra to make observations and verifications but when it came to writing the proof in their worksheets, they resorted to logical arguments.

This may be seen as the phase of transition from verification to proof of the fact that EFGH is a parallelogram. While GeoGebra was instrumental in enabling the students to conjecture and verify that EFGH is a parallelogram, the actual formal proof was possible only when students were able to apply the midpoint theorem to argue that EFGH is a parallelogram. It may be pointed out here that while in a paper-pencil environment the figure is static, in a DGE the vertices of the quadrilateral ABCD may be dragged to enable the student to 'see' that the midpoint theorem actually holds true for any parallelogram EFGH formed by joining the midpoints of a quadrilateral ABCD.

The above exercise highlights the fact that dragging feature of GeoGebra helped students to make conjectures which they might not have been able to make in a paper-pencil approach. Also the mixed strategy of facilitating the exploration of the problem in a DGE along with emphasis on formal proof seemed to work very well.

The result from the 9th graders is excluded, due to page limits.

Conclusions

The study described in this paper based on investigating student's geometrical constructions leading to proof in a DGE is something unusual in both India and Sweden at the compulsory school level. Nevertheless, the research conclusions of both countries seem to be the same. It seems as if students in one class in Gothenburg could have been in Mumbai instead and vice versa without us observing any major differences. One cultural difference we noticed might be that in India the researchers familiarized the students with the basic construction tools in GeoGebra and also facilitated their explorations from time to time whereas in Sweden the students were largely exploring on their own.

The students both in India and Sweden seemed to appreciate this kind of mathematical experience and we as researchers were almost stunned by the speed with which the students picked up the construction possibilities in GeoGebra. Our findings indicate that if given an opportunity to do geometrical constructions in a DGE, even average students in compulsory school will will make conjectures and draw conclusions along the way. We found that students easily use their prior knowledge of geometrical concepts and properties to construct figures in GeoGebra. It was evident that dragging feature of GeoGebra helped students to abstract invariant properties while dynamically changing a figure thus enabling them to make conjectures. For example the 7th graders of both countries dragged the vertices of the outer quadrilateral and conjectured that the inner quadrilateral is always a parallelogram.

Much to our surprise, we found that the process of proving conjectures became a natural extension of the conjecturing for the 7th graders. Since the figures were dynamic and could be changed easily (unlike in paper and pencil), students were quick to make and verify their conjectures. Thus GeoGebra hastened the process and students were able to proceed to formal proof. In the process of writing the proof, however, GeoGebra's role was restricted. Here they converted their observations/verifications (made using GeoGebra) to a sequence of logical arguments which then led to the final proof.

In the 7th grade it appears that student's concept image of a parallelogram was confined to a particular kind of parallelogram but after the midpoints quadrilateral investigation this image was enhanced to include the parallelogram as a result of joining the midpoints of any quadrilateral. Thus according to Vinner's theory the DGE played a significant role in shaping the concept image of the parallelogram. Also viewing the parallelogram in this new light, that is, as a dependent of the outer quadrilateral entails an ontological shift from the earlier concept image of the parallelogram. The stages of interiorization (dealing with and recognizing the quadrilateral as a familiar object) followed by condensation (constructing and manipulating the parallelogram as a separate entity) and reification (recognizing the inner quadrilateral as a parallelogram despite the appearance of the outer quadrilateral) were evident in the explorations by the 7th graders.

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