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EXTENDING NUMBERS WITH NUMBER SENSE

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The paper presents a trajectory for signed numbers involving the context of debt/asset. It is argued that this context in a well-designed trajectory can support both the quantitative thinking and value relationships necessary to extend number sense to this domain. It is argued that the popular contexts of temperature and distance can support order relationships but not the quantitative thinking necessary to extend number sense. Rather than reaching for computer-based contexts, the classic context debt/asset could be used. A strong narrative can help to develop shared intentionality and classroom dialogue involving the tools of bead string and empty number line can help to create a shared model of the situation which can later form a model for thinking about the mathematical relationships of signed numbers.
Signed Numbers, Integers, Number Sense, modelling, empty number line

INTRODUCTION

In this paper we report on a teaching-learning trajectory for signed numbers/integers developed over the last few years. The trajectory developed includes the introduction of fractional numbers also for reasons we do not go into in this paper. The trajectory is an integral part of a decade long process of working with children and teachers to promote conceptual understanding. The approach taken has been informed by both Activity Theory and by the philosophy of Realistic Mathematics Education.

The extension of numbers to include signed numbers introduces the challenge of finding an adequate didactical model. In the case of natural numbers, the sense of what numbers mean has been intimately connected to practices of estimation and quantitative judgement rooted in everyday experiences. Number sense has been also shown to be closely related to the internal image of a number line. The importance of linear representation of numbers shown by Siegler and colleagues also indicates the significance of number line for number sense. We could say that the development of a didactical model for signed numbers faces the challenge of dealing with both these aspects – the quantity aspect and the order aspect.

Although the use of a number line to support the development of number sense in the case of natural numbers is rather recent, it has been different in the case of integers. By the beginning of twentieth century the number line for integers was so well established in school curriculum that F. Klein considered it as 'as a common possession of all educated persons' (Thomaidis & Tzanakis, 2007). Yet difficulties were found with the number line model especially regarding subtraction of negative numbers and led to the search for alternative means for introducing integers.

DISCRETE QUANTITIES

In the sixties authorities such as Dienes, Freudenthal, Gattegno and Sawyer suggested activities with discrete quantities to introduce integers. Yet all these cancellation methods also lead to rules for subtraction of negative quantities that are quite convoluted.

Attempts have been made for children to work out the sign rules themselves, focusing on the sense making activities of children rather than on the physical manipulation of objects. The contexts that have been used have been of different types: dancing couples (Dienes), people leaving bus and boarding bus (Streefland) and people entering and leaving disco hall or teams earning points in a game (Linchevsky & Williams, 1999). In the experiments conducted by Linchevsky and Williams while the children developed intuitive methods for doing addition in one case, the extension to subtraction was not that satisfactory and in the second case although extension to subtraction did take place, but the authors considered it even less satisfactory. They concluded that 'no single model will be 'both intuitive and comprehensive', a conclusion shared by more researchers (Ball, 1993). The focus of these studies was in fact on the development of the rule of signs.

NUMBER LINE

One of the problems with the discrete models is that they do not lay an intuitive basis for the order relationship. At best they can lay a basis for two separate number lines without any natural connection between the two. As discussed by Thomaidis and Tzanakis, the order relationship across the positive and negative integers was also a crucial impetus in the extension of the number system. (2007, p. 5). The number line which makes the ordinal relationships visual can be considered to be a powerful model for making the order relationships in integers/signed numbers transparent.

Due to its ability to reflect a unified order relationship, temperature has been a favoured didactical model. In this case the presence of an arbitrary zero supports the sense of this order relationship showing increasing and decreasing temperature valid across the whole of the number line. Even though negative numbers started their journey firmly linked to the concept of a quantity (whether fictive or 'less than nothing') their final resolution in the formal system was without reference to any quantity and defined in terms of operation with the natural numbers, where only a value relation persisted. Order-focused approaches met difficulty in sense-making as Deborah Ball found during her teaching experiment that it is difficult to make sense of adding the 3 floors above the ground and 4 floors below the ground (1993, p. 381). Schwarz, Kohn and Resnick (1993/94) reviewed the existing real-life models and found them unsuitable and chose a computer-based model.

CONTOURS OF A DIDACTICAL MODEL

It could be argued that any model that needs to be developed for introducing negative numbers should be able to support both quantitative thinking as well as visualization on a number line. We could say along with Resnick and colleagues that for number sense to be developed the model to be followed should fulfil the following criteria. (Schwarz, Kohn and Resnick (1993/94, p 41-42)

Number should refer to two quantities both of which support both unary and binary operations. The negative numbers to be constructed in such a way that the students can extend their sense of positive numbers to them so that they can reason about 'physical or scientific objects that have negative value' with the possibility for 'change', 'combine' and 'compare' to be applicable to negative numbers even as the formal system requirements are taken care of.

The popular contexts of temperature and steps or distances or elevators do lend themselves for unary addition. Temperature of a place can be increased or decreased and one can climb up or down steps. But binary addition is not possible with these models. Being an intensive quantity combining two objects even of the same specific heat would not lead to an 'addition' of their temperatures. With a reference point of zero, temperature can show positive and negative positions, but correspondingly there are no two quantities attached to it. There is only one quantity that increases – average kinetic energy. This would make it an ideal context for developing an intuitive sense of order, but difficult for developing a sense of two quantities, one the additive inverse of the other.

Reviving the classic context - Debt and Asset

We would argue that the classic context of 'debt and asset' is a powerful context and perhaps the only context that can meet the requirements of an adequate didactical model for teaching signed numbers. Earlier studies have shown the positive response of children to the debt and asset context during interviews (Mukhopadhyay et. al, 1990) although this context does seem to have been followed up in teaching experiments. Historically also this was the context that brought forward the need to have negative numbers (apart from algebra) whether in the case of counting sticks of China or Brahmagupta's introduction of negative numbers.

APPROACHES TO DIDACTICAL MODELS – CONTEXTS AND ACTIVITIES

The approaches to developing an adequate model for teaching signed numbers and the use of contexts and activities can be seen in terms of the two types of approaches in mathematics education. One type attempts to find objects that can have a structure similar to the structure of the operations that are to be carried out. In the case of addition and subtractions with numbers up to 100 with carry-over and borrowing, the attempt was to find objects that can exemplify the relationships between the different place-value columns. A good example of this is the prescriptive use of Dienes blocks in which first the smaller blocks are added as is done in the written algorithm without considering the spontaneous tendency to first add the larger units. Although less alienating than the earlier formal approaches, the problems with such structuralist approaches have been already brought out by many authors (Gravemeijer K., 1994) (Treffers A., 1991).

The other type starts from contexts and situations in which children experience meanings that were analogous to the mathematical ideas that were to be learnt: quantity, increase, take way and so forth. Within this group at present we can discern two major streams regarding the nature of the relationship between the familiar meaningful context of the child and the concept to be learnt. One stream generally identified as constructivist, in fact the dominant one, takes to exposing the child to as many different everyday contexts as possible.

The other stream mainly represented by Realistic Mathematics Education (RME), considers a special well-designed context as the basis for developing understanding. Through a series of level-raisings with increasing abstraction a (mature) understanding of the concept is reached. (Treffers A., 1991).

These two approaches can again be seen to be related to different understandings about the nature of a concept; One assumes a concept to be a generalisation from a large number of examples, in fact the classical idea of a concept. The second considers model building to be the main activity in the development of a concept and the model to emerge from a typical context.

“In the opinion of the realists, on the other hand, it is especially by means of strong models that children are given the opportunity to bridge the gap between informal, context-bound work and the formal, standardised manner of operation, through the constructive contribution of the children themselves. In short they take the position that assistance can be given via the presentation of fitting models and schemes, that direction can be given to learning, that something essential can be offered from ‘outside’ ” (Treffers A.,1991, p. 33).

RME in this sense is different from constructivist or socio-constructivist approaches. In the RME tradition this model building activity has been put forward as a series of steps. Models are expected to meet very stringent conditions and that they must be ‘models of’ context problems which will serve as ‘models for’ the pure subject restricted and applied arithmetic in the pertaining area.’ (Treffers A., 1991, p. 33). Gravemeijer has further differentiated the ‘model of’ and ‘model for’ views as below with the ‘model of ‘being the ‘referential level, where models and strategies refer to the situation which is sketched in a problem’ and the ‘model for’ being the ‘general level, where a mathematical focus on strategies dominates the reference to the context’ (Gravemeijer K., 1994, p. 101)

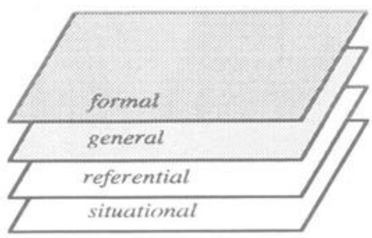

Figure 1: Four Levels of Progressive Mathematisation (Gravemeijer K. , 1994, p. 102)

This role of the model in RME can be considered to be consistent with the Vygotskian perspective where learning is mediated by tools that include inscriptions, words (generalisations) and models and where the concepts go through a process of evolution. It also relates to the idea of learning involving a transition from the inter-mental to the intra-mental plane. Vygotsky posed the question of the relationship between everyday concepts and scientific concepts, and one of the biggest challenges in teaching can be considered to be the establishment of this relationship. Vygotsky distinguishes scientific concepts by the fact that they are embedded in a structure where concepts are related to concepts through 'supra-empirical connections between concepts' (Vygotsky, 1987/1934, p. 234) The RME perspective of a 'model of' and 'model for' can be considered as an option for the transition from everyday concepts rather the option of verbal definitions that Vygotsky proposed.

Developing Shared Intentionality

The development of an adequate didactic model for teaching of negative numbers can be seen in terms of the dynamics of establishing shared intentionality (Tomasello & Carpenter, 2007) in the classroom. This process of developing a shared intentionality has been characterised as appropriation by Leontyev to distinguish it from adaptation (Leontyev, 1981, p292-301). This is a process that goes through various stages as Vygotsky has argued even in the case of material objects with a clear referent. This learning is mediated by tools that include inscriptions, words (generalisations) and models and where the concepts go through a process of evolution. In the designing of didactical models to achieve intersubjectivity, the RME (Realistic Mathematics Education) perspective has been found to be very useful.

DEVELOPING A TEACHING-LEARNING TRAJECTORY

The first steps for the trajectory for teaching signed numbers involving loan and asset emerged between 2005 and 2007 and the first part of the trajectory was implemented in 2009. The outline of the trajectory was shared with teachers in a school and they incorporated it into their curriculum for integers in Grade VI (11 year olds) for the last two years involving about 150 children. The teachers reported that the students enjoyed the story and continued to use it to give meanings to expressions. But they reported that children did not make use of the number line easily. This contributed to the decision to take a new complete set of classes (15 classes) which were conducted during October – November 2011.

Following are the aspects of the trajectory in its two main phases of developing a 'model of'' a context and a 'model for' doing mathematical operations.

'MODEL OF' A SITUATION

The anchoring context from which a model is to be generated is presented in the form of a story to children.

The narrative concerns a boy called Bunty who decides to sell vegetables after school to help his mother, since his father had fallen ill. Children are told that Bunty decides to put his money in a box and give it to his mother when it becomes Rs.100. He also decides to buy vegetables everyday for Rs.200. He decides to hang a string of beads in the room to know how much money he had in the box. At this juncture the 100 bead Ganit Mala (Bead String with alternating colours of ten beads) is introduced by the teacher and hung in the class. It is told that on the first day Bunty sells vegetables worth Rs.250 and children come forward to put a clip at 50 showing how much money Bunty has in the box after keeping Rs.200 for buying vegetables for the next day. The next day Bunty has only Rs.30 to add and children shift the clip to 80 on the Ganit Mala.

Then starts a few days of mishaps: one day a cow eats the vegetables when Bunty goes to get water, the next day it rains heavily and so on and Bunty has only Rs.10 in his box. This narrative which was done in full story-telling format had all the children engaged. It continued somewhat as given below, although there is no transcript of the class and I was the teacher.

Teacher: Next day also was a bad day for Bunty. It rained a lot and very few people came to buy the vegetables. By the evening many of the vegetables got spoilt and he earned only Rs.170 that day. He was in a quandary about what to do. There were only ten rupees left in his box. Then he remembered that the uncle who sells samosas (a popular snack) nearby had said that he could lend some money. Therefore he borrowed…

Children completed the sentence to say, "Twenty rupees."

Teacher: That is right. But tell me where do you think Bunty put the clip?

Some children answer to say, "Zero!"

Teacher: It is true that there was no money in the box. But Bunty was not happy to put the clip at zero, because he wanted to be reminded that he had less than zero rupees with him because he had to return twenty rupees to the samosewala uncle. So he thought for a while and he made a special Ganitmala. Can any of you guess what was the kind of Ganitmala that Bunty made?

Some children tried to give an answer which was shared in the class. Then the teacher takes out the Integer Ganitmala (Figure 2) in four colours.

Teacher: See, this is the kind of special Ganitmala that Bunty made to show how much money he actually had. Who can say where Bunty would have put his clip now to show that he had to pay Rs.20 to samosewala uncle?

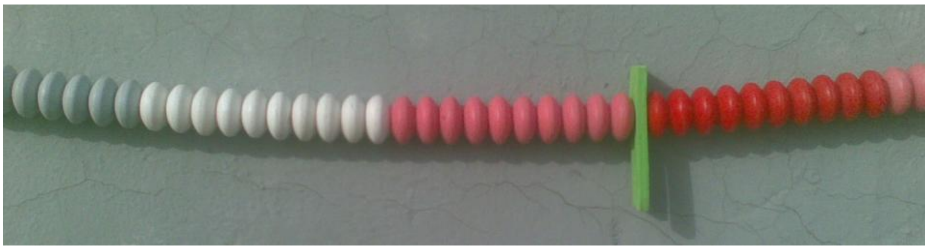

Figure 2: Bunty has Rs.10 in the box

Now about three or four of the children in the class could immediately think of how it could be shown on the new Ganitmala and it took very little time before the whole class could make sense of how to use the new Ganitmala to show the money that Bunty had.

In a similar vein subtraction of a negative quantity is also introduced.

Teacher: Things had not gone very well for Bunty and now he had to pay a total of Rs.80 to samosewala uncle. One day uncle saw Bunty sitting very sad and lost in his thoughts. He knew what the boy was thinking about. He did not want Bunty to get disheartened with his new enterprise also at the same time he did not want Bunty to get spoilt. He told him, "*Are, beta, pareshan mat ho jao.*

Mein tumara aadha loan maaf kardunga. (Oh! Son, Don't' worry! I will write-off half your loan)". Bunty did not want to accept that but samosewala uncle was very insistent and finally Bunty decided to accept it.

The response of the children to the above narrative has been phenomenal and the teachers have reported about how some children refer back to this context to solve their equations even later.

Peled and Carraher report an instance of using a number line with the context of money and signal the difficulties that might emerge. In the case reported, the number line is represented by a clothesline with labels for the numbers from -10 to +20 hanging from the line. In the situation enacted in the reported classroom, one of the children has taken loan of $2 and the discussion is about where to hang the clip. The transcript of the class shows that there is confusion about where to hang the clip at +2, 0 or -2. Peled and Carraher conclude to say that many children and even adults 'tend to take into account only the tangible assets' and therefore advises taking up money contexts with caution.(Peled & Carraher, 2006, pp. 312-316). Their suggestion is to avoid the everyday arithmetic contexts and to consider algebraic contexts related to temperature or to travel.

The Bunty context and money and number line context that Peled and Carraher sketch are different in many ways. We could say that the differences are mainly of two types. One is related to the design of the situation itself where the script needs to be carefully constructed. The other is related to the design but involves a general aspect about the process of constructing the situation model. In the constructivist climate of U.S.A. it is expected that children would construct the model, while what is expected in the Bunty script is that children are able to' live in and experience as sensible' the choices being made. The question is put in terms of a dilemma of Bunty who does not want to put the Clip at zero even though there is no money in the box. And the idea of extending it to the other side to show how much he owes is not difficult for children to internalise. This script has been has been so far used in this manner in twenty class rooms over the last three years and the teachers have not reported any difficulty for children in recognising the position.

Placing Bunty's positions on the Empty Number Line

Later Bunty's situation is marked on an empty number line as below (Figure 3). The children in the schools were this module has been tried out have been using the Jodo Gyan mathematics curriculum for Grade 1 and were familiar with the use of the empty number line and Day 1 to 3 were marked easily. As Day 4 has to be marked there is a discussion in the class about how the number is to be noted and the class agrees that it can be notated with a raised minus sign. Following this Day 1, 2 and 3 are also marked with a raised plus sign.

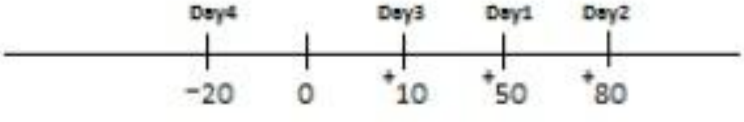

Figure 3: Bunty's Position on the Number Line

Once the days are marked on the ENL, there is a discussion about which day Bunty had the greatest wealth and which day he had the least wealth. This is an extremely important aspect. Through this conversation the order relationships are being established. The words **greatest** and **least** are also very consciously used by the teacher. The directions of increasing wealth and decreasing wealth are also clearly established in the class. Raised sign for positive and negative numbers – a semiotic tool to distinguish from the signs for operation – is introduced during this transition to number line to compare the wealth positions on different days. (see Vlassis, 2004; for children's interpretations of the minus sign)

Consolidating distinction between wealth, cash and loan

Based on the experience of the set of classes conducted in 2011, the story was modified to develop a shared understanding in the class of the meaning of the clip on the bead string. This is done in the context of Bunty's sister who comes to Bunty to ask him whether he could give her Rs.20 to buy a notebook and thinks that he cannot give her the money because she sees his clip at positive 10. But then he surprises her to say that he can give her Rs.20 because he has Rs. 50 in the cash. He explains that he had put the clip at 10 because he had to return the loan of Rs.40 to *samosewala* uncle. Following this the class discusses about what could the different interpretations for the clip at positive 10 and negative 10 and so forth.

TOWARDS 'MODEL FOR' MATHEMATICAL OPERATIONS

Later activities are done on based on the model of the empty number line which was earlier a part of the situation of Bunty. The term 'value' of a number is introduced in the case of positive and negative numbers to show the order relationship between the numbers reflecting a similar relationship when one considered the numbers as representing the position of Bunty's wealth. In consonance with the earlier situation specific model it is seen that the value of a number increases in going from left to right and the terms 'greater value' and lesser value are used and not bigger and smaller. The terms 'bigger' and 'smaller' are reserved to indicate the size of the numbers independent of the sign and connected to absolute value.

A crucial activity that was introduced this year related to considering a unary operation with positive and negative quantities and considering them in terms of the impact on value. During whole class discussion, the following is brought out where the symbols are spoken as increase/decrease of positive/negative twenty.

$+{}^{+}20$ value increases

$-{}^{-}20$ value increases

$+{}^{-}20$ value decreases

$-{}^{+}20$ value decreases

It was felt that various activities on the empty number line similar to what is done in the case of numbers up to 100 to develop number sense (Menne, 2001, Menon, 2012) needs to be also done in the case when the number line gets extended. The feedback to these type of activities has been positive and further modifications would be made in the next round of interventions.

An example is the activity of going from one number to another. Unlike in the case of natural numbers, going from one number to another is shown in two ways involving either addition or subtraction using the convention of above the number line for addition and below the number line to show subtraction as is done in the case of early numbers also.

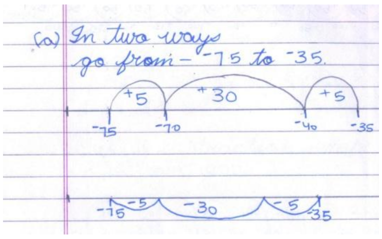

Figure 4: Jumping from one number to another

Assessment results indicate that children are able to use the number line as a tool to think about signed numbers. All the 22 children were able to locate numbers appropriately with up to 3 digit numbers and more than 20 could do it for larger numbers also. 70 to 80 % of the children were able to move on the number line in more than one way using the convention. About 65% of the students used 0 as a landmark in this movement on the number line. Children also used the number line to solve problems. Problems such as 25-35 were done correctly by more than 80% and -30-(-55) by about 73% of the students.

EXTENDING NUMBER SENSE

The debt/asset context appears to be able to function as an anchoring context to think about signed numbers in terms of a net effect of two quantities. It is suggested that such a sense of quantity and order are both necessary to grasp signed numbers with number sense. The classroom experience indicates that with suitable semiotic mediating devices, children can go on to reach further levels of abstraction to appropriate the cultural tool of number line to think about signed numbers. This second aspect is to be further developed in the next round of interventions.

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